SOME RENEWAL THEOREMS WITH APPLICATION TO A FIRST PASSAGE PROBLEM[1]

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1. Introduction. Let X_i, $i=1, 2, 3, \cdots$ be a sequence of independent and identically distributed random variables with $E|X_i| < \infty$, $EX_i = \mu > 0$. Write $X_i^- = -\min(0, X_i)$, $S_n = \sum_{i=1}^n X_i$ and $M_n = \max_{1 \leqq k \leqq n} S_k$. In this paper we shall discuss the asymptotic behaviour as $x \to \infty$ of the sums $\sum_{n=1}^\infty a_n \Pr(S_n \leqq x)$ and $\sum_{n=1}^\infty a_n \Pr(M_n \leqq x)$ for certain classes of positive coefficient sequences $\{a_n\}$ and use the results on the latter sums to investigate the behaviour of the first passage time out of the interval $(-\infty, x]$ for the process S_n as $x \to \infty$.

The analysis that we shall use in obtaining the theorems on asymptotic behaviour follows closely on that of Smith [6] who discussed sums $\sum_{n=1}^\infty a_n \Pr(S_n \leqq x)$ for a class of coefficient sequences that we shall also discuss and for non-identically distributed random variables. In fact, our Theorem 1 follows directly from a specialization of the analysis of Smith. One of the particularly interesting characteristics of this technique is that it enables us to study the asymptotic behaviour of the sums $\sum_{n=1}^\infty a_n \Pr(S_n \leqq x)$ and $\sum_{n=1}^\infty a_n \Pr(M_n \leqq x)$ in the one operation in spite of essential differences in their behaviour.

2. Renewal theorems. For the first set of positive term coefficient sequences $\{a_n\}$ that we consider we shall suppose (as in [6]) that there exist real numbers $\alpha > 0$, $\gamma \geqq 0$ and some non-negative function of slow growth $L(x)$ such that

$$(1) \qquad \sum_{n=1}^\infty a_n x^n \sim [\alpha/(1-x)^\gamma] L(1-x)^{-1}, \qquad \text{as} \qquad x \to 1^-.$$

This is satisfied, for example, if

$$a_n \sim [\alpha/\Gamma(\gamma)] n^{\gamma-1} L(n) \qquad \text{as} \qquad n \to \infty$$

using an Abelian theorem of Doetsch [3], 460.

In the subsequent work we shall need the following definition:

Definition. The index k of the sequence $\{a_n\}$ is the least real k such that $a_n = O(n^k)$.

Consideration will be restricted to cases where $\sum a_n$ diverges.

Theorem 1. *Suppose* $E|X| < \infty$, $EX = \mu > 0$. *Let k be the index of the sequence $\{a_n\}$ and m be non-negative. In order that*

$$\sum_{n=1}^\infty a_n \Pr(S_n \leqq x) \sim [\alpha L(x)/\Gamma(1+\gamma)](x/\mu)^\gamma \qquad \textit{as} \qquad x \to \infty$$

for each sequence $\{a_n\}$ such that $k \leqq m$ it is necessary and sufficient that $E|X^-|^{m+2} < \infty$.

Received 2 June 1965; revised 6 January 1966.

[1] Research supported in part by the National Institutes of Health.