A THEOREM ON THE GALTON-WATSON PROCESS[1]

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In this note we will prove a theorem concerning a limiting distribution associated with the Galton-Watson process. Specifically, we consider a stochastic process, $\{Z_n; n = 0, 1, \cdots\}$, with the following properties:

(1) $Z_0 = 1$;

(2) if P denotes the probability measure associated with the process, then $P(Z_1 = i) = p_i, i = 0, 1, \cdots$. Moreover the process is a Markoff process with transition probabilities,

$$P_{ij} = P(Z_{n+1} = j \mid Z_n = i) = \sum_{k_1+k_2+\cdots+k_i=j} p_{k_1} \cdot p_{k_2} \cdots p_{k_i},$$

$$i = 1, 2, \cdots, j = 0, 1, \cdots, P_{0j} = 0, j = 1, 2, \cdots, \quad \text{and} \quad P_{00} = 1;$$

(3) $p_i \neq 1$ for all i; and

(4) $E(Z_1) = m > 1$.

We will show that the random variables, (Z_n/m^n), $n = 0, 1, \cdots$, converge a.e. to a random variable, W, whose probability distribution has a jump at the origin and a continuous density function on the set of positive real numbers. Levinson and Harris have proved similar theorems but under more restrictive assumptions and by using quite different arguments. Specifically, Levinson [4] by assuming that $E(Z_1 \log Z_1) < \infty$ and Harris [3] by assuming that $E(Z_1^2) < \infty$ have established our result. Harris has also proved that his assumptions imply convergence in the mean of the (Z_n/m^n)'s, and both Harris and Levinson have proved that their assumptions imply that $E(W) = 1$ and that $P(W = 0) = q < 1$, where q is a number to be defined later. In contrast under our assumptions we can only prove that $E(W) \leqq 1$ and that $P(W = 0) = q$ or 1. However we will show in a forthcoming paper with Harry Kesten that if Assumptions 1 through 4 hold and if $P(W = 0) = q$, then $E(W) = 1$. Moreover $E(W) = 1$ only if $E(Z_1 \log Z_1) < \infty$.

The probability generating function of Z_1 will be denoted $f(s)$ and is defined by the equation, $f(s) = \sum_{k=0}^{\infty} p_k s^k$, on the set of all complex numbers s such that $|s| \leqq 1$. The probability generating function of Z_n will be denoted by $f_n(\cdot)$. We will make repeated use of a few facts about the $f_n(\cdot)$'s that are stated briefly below.

(5) $f_{n+k}(s) = f_n(f_k(s)) = f_k(f_n(s))$.

(6) There exists a unique real number q such that $0 \leqq q < 1, f(q) = q$.

(7) For all $s \,\varepsilon\, [q, 1)$ we have $1 > s \geqq f(s) \geqq f_2(s) \geqq \cdots \geqq f_n(s) \geqq q$ with

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