ON A THEOREM OF CRAMÉR AND LEADBETTER[1]

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1. Introduction. In a recent paper [1], Cramér and Leadbetter have given an integral formula for the kth factorial moment of the number of upcrossings of the zero level by a stationary Gaussian process in unit time. More specifically, suppose $X(\cdot)$ is such a process and that $X'(\cdot)$ exists and has continuous sample paths. If N is the number of upcrossings of zero by $X(\cdot)$, there is a positive function f defined on the unit cube Ω in k-space and dependent on the joint densities of $X(\cdot)$ and $X'(\cdot)$ for which $EN(N-1)\cdots(N-k+1) = \int_\Omega f\,d\mu$, μ Lebesgue measure. From this result, one may also give expression to the kth factorial moment of the number of zeros of $X(\cdot)$.

A second fact is established in [1], viz., $\int_\Omega f\,d\mu \leqq EN(N-1)\cdots(N-k+1)$ even if $X'(\cdot)$ has discontinuous sample paths. Consequently, the formula still holds provided f is not integrable. At present, the relationship between (a) f integrable and (b) $X'(\cdot)$ has continuous sample paths is not known.

In this paper we find for quite general processes, a particular submartingale (relative to the Lebesgue measure space) sequence $\{f_n\}$ of functions on Ω for which $\int_\Omega \liminf f_n\,d\mu \leqq EN(N-1)\cdots(N-k+1)$. Under suitable conditions, $f_n \to_{\text{a.s.}} f$ and $\int_\Omega f\,d\mu = EN(N-1)\cdots(N-k+1)$. As a special case, this is shown to hold for the processes of [1] without the continuity restriction on $X'(\cdot)$ (to achieve this, $X(\cdot)$ is subjected to a nondegeneracy requirement also required in [1]).

2. Moments of upcrossings. Let $X(\cdot)$ be a separable stochastic process on the unit interval. We assume throughout that $X(t)$ has a continuous distribution for each $t \in [0, 1]$ and that $X(\cdot)$ has continuous sample paths with probability 1.

The number N of upcrossings of zero by the process $X(\cdot)$ is approximated by counting those of a polygonal process tied to $X(\cdot)$ at points of the form $i/2^n$. Specifically, let U_{ni} be the indicator of the event $\{X[(i-1)/2^n] < 0 < X(i/2^n)\}$. Under the above assumptions, $\sum' U_{ni_1}\cdots U_{ni_k} \uparrow N(N-1)\cdots(N-k+1)$ a.s., where $\sum'$ denotes summation over all appropriate $\mathbf{i} = (i_1, \cdots, i_k)$ having distinct entries (cf [1]). Then, according to the monotone convergence theorem,

$$(1)\quad EN(N-1)\cdots(N-k+1) = \lim_{n\to\infty} \sum{}' P\{X[(i_1-1)/2^n] < 0 < X(i_1/2^n), \cdots, X[(i_k-1)/2^n] < 0 < X(i_k/2^n)\}.$$

We will subsequently consider as an auxiliary probability space the unit cube Ω in k-space, together with the Lebesgue measurable subsets and Lebesgue meas-

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