# FIDUCIAL THEORY AND INVARIANT ESTIMATION[1]

By R. B. Hora[2] and R. J. Buehler

*University of Minnesota*

**1. Introduction.** A class $\Omega$ of distributions may be called invariant under a group $\mathcal{G}$ of transformations of the sample space $\{x\}$ if $\Omega$ is closed under $\mathcal{G}$; that is, whenever $x$ has a distribution belonging to $\Omega$, so does $gx$ for any $g \varepsilon \mathcal{G}$. In a fairly large body of decision theoretic literature, invariance enters into criteria for solutions, or is used as a tool. Fiducial theory is also intimately connected with invariance theory. In the present paper some links will be established between the decision theoretic and fiducial viewpoints.

Fisher (1934) gave fiducial distributions which he considered to be appropriate for location and scale parameters. These solutions were studied in detail by Pitman (1939), who showed, among other things, how certain "best" estimators could be defined in terms of expectations with respect to fiducial distributions. In the terminology of decision theory, Pitman's estimators are "best invariant" or "minimax invariant" estimators. Fraser's (1961a), (1961b) group-theoretic approach to fiducial theory, using Haar measures, is useful in providing a precise mathematical framework which is consistent with Fisher's in the case of location and scale parameters, and apparently in most other cases as well.

The present paper shows how certain of Pitman's results can be generalized using Fraser's theory. The results in Section 5 on "best" invariant estimators defined by means of fiducial distributions could actually be formulated and proved without reference to fiducial theory. For example, Blackwell and Girshick (1954), p. 314, express the best estimator of a location parameter in terms of the conditional expectation of the first observation $x_1$ given the differences $x_2 - x_1, \cdots, x_n - x_1$. Thus fiducial theory may be regarded as a convenient, but not essential, tool for obtaining desirable estimators.

In Section 2, assumptions similar to Fraser's on the class of distributions are spelled out in detail. The equivalence of the fiducial distribution to a posterior distribution is pointed out. Theorem 2.1 establishes the identity $E_f^x H = E_a^\omega H$ where $H$ is an invariant function of the observations and parameters, $E_f^x$ denotes expectation with respect to the fiducial distribution given $x$, and $E_a^\omega$ denotes conditional expectation given any value of the ancillary statistic. The theorem is not restricted to location and scale parameter cases.

Section 3 gives four examples of location and scale parameter families whose generality increases from the case of one location parameter to that of two location and two scale parameters. The latter includes the Behrens-Fisher problem

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[2] Now at Purdue University, Indianapolis, Indiana.