BOUNDS ON THE DISTRIBUTION FUNCTIONS OF THE BEHRENS-FISHER STATISTIC[1]

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1. Introduction. It is commonly accepted in the case of two independently distributed normal variables that the distribution function of the Behrens-Fisher statistic is bounded, for all values of the variance ratio σ_1^2/σ_2^2, by the distribution functions of the Student-t variates with $(n_1 + n_2 - 2)$ and min $(n_1 - 1, n_2 - 1)$ degrees of freedom (df). By the Behrens-Fisher statistic we mean

$$V = [\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)]/(s_1^2/n_1 + s_2^2/n_2)^{\frac{1}{2}}$$

where $x_{11}, \cdots, x_{1i}, \cdots, x_{1n_1}$ and $x_{21}, \cdots, x_{2j}, \cdots, x_{2n_2}$ are samples from the two independent Gaussian distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ respectively and

$$\bar{x}_i = \textstyle\sum_j x_{ij}/n_i ;$$

$$s_i^2 = \textstyle\sum_j (x_{ij} - \bar{x}_i)^2/(n_i - 1).$$

The purpose of this note is to supply an analytical proof of the above proposition.

This result has certain implications. If a critical value for the Behrens-Fisher statistic is specified that is constant for all ratios of the observed sample variances, then it should lie between those of the Student-t variates with $(n_1 + n_2 - 2)$ df and with min $(n_1 - 1, n_2 - 1)$ df at the desired level of significance. In the "equivalent degrees of freedom" approaches, it is reasonable that $(n_1 + n_2 - 2)$ and min $(n_1 - 1, n_2 - 1)$ be bounds on the number of degrees of freedom with which to enter the Student-t table; also we may then put limits on the tail probability. However a constant critical value is not desirable in this problem and effective use of prior knowledge may yield critical values which are not bounded by those of the Student-t variate with $(n_1 + n_2 - 2)$ and min $(n_1 - 1, n_2 - 1)$ df [1].

2. Development. A formal statement of the basic proposition is as follows:

Theorem. *Let X be normally distributed with zero mean and unit variance, and let f_1W_1 and f_2W_2 be distributed as chi-square variates with f_1 and f_2 degrees of freedom* (df) *respectively, such that X, W_1, and W_2 are mutually independently distributed. Then for all γ in the interval* $0 \leqq \gamma \leqq 1$,

$$P\{|T_1| < v\} \leqq P\{|V_\gamma| < v\} \leqq P\{|T_2| < v\} \tag{1}$$

where

$$V_\gamma = X/(\gamma W_1 + (1 - \gamma)W_2)^{\frac{1}{2}}$$

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