J. Kubilius, *Probabilistic Methods in the Theory of Numbers* (translated from the Russian by Gretchen Burgie and Susan Schuur). Vol. 11 of Translations of Mathematical Monographs, American Mathematical Society, Providence, 1964. xviii + 182 pp. $8.60 ($6.45 to members).

Review by W. J. LeVeque

University of Michigan

The scope of this monograph is not quite so wide as the title indicates. The author has made no attempt to cover all the applications of probability to number theory, but has restricted himself to one topic: the probabilistic analysis of the behavior of additive number-theoretic functions. There is no mention, for example, of the application of probabilistic methods in the metric theory of Diophantine approximations, or in the theory of distribution modulo 1.

By an additive number theoretic function, one means a function defined on the positive integers and such that $f(mn) = f(m) + f(n)$ whenever m and m are relatively prime. If in addition $f(p^\alpha) = f(p)$ for every prime p and positive integer α, then f is said to be strongly additive. It follows immediately from the unique factorization theorem that if f is additive, then $f(n) = \sum_{p^\alpha \parallel n} f(p^\alpha)$, and that if f is strongly additive then $f(n) = \sum_{p|n} f(p)$. (Here $p \mid n$ means that p divides n, and $p^\alpha \parallel n$ that p^α is the exact power of p occurring in the factorization of n.) If $\delta_p(m)$ is 1 or 0 according as $p \mid m$ or not, the latter equation can be written in the form

$$(1) \qquad f(m) = \textstyle\sum_p \delta_p(m) f(p).$$

Now let $D\{a_n\}$ be the asymptotic density of the sequence $\{a_n\}$ of positive integers. If A and B are arithmetic progressions of moduli d and d', respectively, then $DA = d^{-1}$, $DB = d'^{-1}$, and if $(d, d') = 1$, then the Chinese remainder theorem implies that $D(A \cap B) = DA \cdot DB$. In particular,

$$(2) \quad D\{m: \delta_{p_1}(m) = \epsilon_1, \cdots, \delta_{p_r}(m) = \epsilon_r\} = \textstyle\prod_{j=1}^{r} D\{m: \delta_{p_j}(m) = \epsilon_j\},$$

where each of $\epsilon_1, \cdots, \epsilon_r$ is 0 or 1, and $p_1, \cdots, p_r$ are distinct primes. Roughly speaking, the basis of the present book is that, although density is not a countably additive measure, it should be possible to utilize the independence assertion (2) to analyze the behavior of f as a sum (1) of independent random variables. The difficulty with additivity can be temporarily circumvented by truncating the sum in (1), and considering the function $f(m)_r = \sum_{p<r} \delta_p(m) f(p)$, to which the standard limit theorems apply, but there is then a very delicate interchange of limit processes required to obtain information about $f(m)$ itself. This interchange depends, as usual, on obtaining uniform estimates for certain error terms; the latter are provided in this case by a (Brun or Selberg) "sieve" argument.

Chapter I of the present book is devoted to the basic arithmetic lemmas that will be needed later. These include estimates for certain sums in which the index