K. ITO AND H. P. MC KEAN, JR., *Diffusion Processes and their Sample Paths*, Academic Press, New York and Springer-Verlag, Berlin, 1965. xvi + 321 pp. $14.50.

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This is the first systematic account of the theory of Brownian motion and its extensions and applications since the appearance of P. Lévy's monograph twenty years earlier. The subject then seemed exciting and important as an intuitive model of many phenomena in semigroups, differential equations and potential theory. Today this is no longer a mere article of faith among Brownian motion enthusiasts, but rather the result of a unified theory which cuts across the borderlines of formerly separate areas of analysis. Therefore the organization of the material for this book was a non-trivial task which the authors accomplished with skill and judgement. In outline, the book consists of three parts. One-dimensional Brownian motion is treated in Chapters 1–2, general one-dimensional diffusion in Chapters 3–6, and in Chapter 7 classical potential theory is developed via the theory of n-dimensional Brownian motion.

The book begins with random walk on the integers (a good opportunity, even for the expert, to learn the notation), introduces Brownian motion, its zeros, continuity properties, absorption probabilities, and Chapter 1 terminates with F. Knights approximation of Brownian motion by random walks, which implies Donsker's invariance principle. Chapter 2 further develops the sample function properties, with emphasis on Lévy's local time, Trotter's theorem and recent refinements. Chapter 3 introduces one-dimensional diffusion (defined as a real valued, continuous, strictly Markovian process), decomposes its state space into intervals with singular end points, and introduces Dynkin's useful infinitesimal generator (in $(T_t - I)/t$ the time t is replaced by the expectation of a family of absorption times which tend to zero). In Chapter 4 these generators are explicitly calculated, resulting in the classification of all possible diffusions according to their speed measure and scale, which in turn determine the generator. The generators are shown to be generalized second derivative operators with respect to the speed and scale—this is the fundamental result of Feller, whose diffusion theory is here simplified by full use of the probability interpretation of the speed and scale. In Chapter 5 the principal result, which here appears for the first time, is that ordinary Brownian motion can be transformed into any diffusion whatever, by a suitable random time change which depends on the local time studied in Chapter 2. This beautiful result has many interesting applications. In particular it permits the complete solution of old problems concerning the most general Brownian motion with elastic barriers, which in part served as the motivation for Feller's diffusion theory. Finally Chapter 6 is devoted to the fine structure of the sample paths of the general linear diffusion.