

NOTES

THE GROWTH OF A RECURRENT RANDOM WALK¹

BY CHARLES STONE

University of California, Los Angeles

Let X_1, X_2, \dots denote independent identically distributed non-degenerate random variables and set $S_n = X_1 + \dots + X_n$. The random walk S_n is called *recurrent* if for some number M $P(|S_n| \leq M \text{ for some } n \geq m) = 1$ for all m . The purpose of this note is to prove the following

THEOREM. *If S_n is recurrent, then*

$$(1) \quad P\{\limsup_{n \rightarrow \infty} S_n/n^{\frac{1}{2}} = \infty\} = 1.$$

The author was led to this result by a conjecture of Y. S. Chow, namely the **COROLLARY.** *If $E|X_k| < \infty$ and $EX_k = 0$, then (1) holds.*

It is plausible that (1) also holds whenever $P\{S_n \geq 0 \text{ for some } n \geq m\} = 1$ for all m .

PROOF OF THEOREM. Let μ denote the common distribution of the X_k 's. Then we can find positive constants a and b and probability measures ν and φ such that $\mu = a\nu + b\varphi$ and φ is non-degenerate and has zero first moment and finite second moment σ^2 . Let $\hat{\mu}$ and $\hat{\nu}$ denote the characteristic function of μ and ν respectively. A direct computation shows that there exist positive constants ϵ , c_1 and c_2 such that

$$(2) \quad 0 \leq c_1 \Re(1 - r\hat{\mu}(\theta))^{-1} \leq \Re(1 - r\hat{\mu}(\theta))^{-1} \\ \leq c_2 \Re(1 - r\hat{\mu}(\theta))^{-1}, \quad |\theta| \leq \epsilon \text{ and } 0 < r < 1.$$

Let Y_1, Y_2, \dots be independent random variables with common distribution ν and set $T_n = Y_1 + \dots + Y_n$. It follows from (2) and Theorem 3 of Chung and Fuchs [1] that T_n is a recurrent random walk. (If $E|X_k| < \infty$ and $EX_k = 0$ then this fact follows alternatively from Theorem 4 of [1]). Let Z_n be independent random variables with common distribution φ and set $U_n = Z_1 + \dots + Z_n$. Let ξ_1, ξ_2, \dots be independent identically distributed random variables such that $P\{\xi_k = 1\} = a$ and $P\{\xi_k = 0\} = b = 1 - a$, and set $j(n) = \xi_1 + \dots + \xi_n$ and $k(n) = n - j(n)$. We can assume that the Y_k 's, Z_k 's, and ξ_k 's are mutually independent.

Set $V_n = T_{j(n)} + U_{k(n)}$. Then V_n has the same probabilistic structure as S_n . By the zero-one law for independent random variables, in order to obtain (1) it suffices to show that for any N

$$(3) \quad \lim_{m \rightarrow \infty} P\{V_n/n^{\frac{1}{2}} \geq N \text{ for some } n \geq m\} > 0.$$

Received 23 February 1966.

¹ The preparation of this paper was sponsored in part by N.S.F. Grant GP-5224.