A LIMIT THEOREM FOR PASSAGE TIMES IN ERGODIC REGENERATIVE PROCESSES

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1. Introduction. In a previous note, "A Technique for Discussing the Passage Time Distribution for Stable Systems" (Keilson, 1965) it was pointed out that for an ergodic Markov diffusion process or birth-death process on a state space having an inaccessible boundary, the distribution of passage times to states of low probability was approximately exponential, and was asymptotically exponential for any sequence of states approaching the inaccessible boundary. It was stated that this well known limiting exponential behavior was to be expected for a much broader class of ergodic processes with inaccessible states, but no explicit results were given. In this note, a theorem will be presented demonstrating such behavior for any ergodic regenerative process in continuous time.

The reader is reminded that a regenerative process in continuous time (which may or may not be Markov) is a temporally homogeneous process $\mathbf{X}(t)$ on an abstract state space of elements $\mathfrak{x}$ characterized by an imbedded sequence of regenerating events having a positive probability of recurrence. (The term event is used here to mean an occurrence in time, with zero duration in time, such as an arrival to or departure from a given set of states. For a given regenerative process, there may be many classes of regenerating events available and attention will focus on some single specified class.) Each such regeneration destroys the "memory" of a process sample, i.e., the statistical behavior of a sample subsequent to such a regeneration is independent of the history of the sample before the regeneration. The time intervals separating successive regenerations then constitute a sequence of independent identically distributed positive random variables. When these have a finite expectation, a renewal process may be associated with the regenerations, and the regenerative process $\mathbf{X}(t)$ is then ergodic, i.e., for any subset A of $\mathfrak{X}$ and any initial state $\mathbf{z}$, $\lim_{t\to\infty} P(\mathbf{X}(t) \, \varepsilon \, A \mid \mathbf{X}(0) = \mathbf{z})$ is a non-degenerate measure $P_\infty(A)$ independent of $\mathbf{z}$. For an extensive discussion of such regenerative processes see W. L. Smith (1955), and J. F. C. Kingman (1964).

It will be convenient and will entail no loss of generality to regard the process $\mathbf{X}(t)$ as being a multivariate Markov process, with the number of random variables finite or infinite, and to focus attention on the regeneration events associated with arrival to, departure from, or passage through some particular state $\mathbf{x}_0$ in the state space $\mathfrak{X}$. We will be interested in a sequence of passage time distributions $F_N(x)$ defined in the following way. For each value N there is given a decomposition $\mathfrak{x} = \mathfrak{x}_1^{(N)} + \mathfrak{x}_2^{(N)}$ of the state space into two disjoint subspaces

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