A LOCAL LIMIT THEOREM FOR A CERTAIN CLASS OF RANDOM WALKS[1]

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1. Introduction. In [3] Karlin and McGregor considered random walks on the non-negative integers of the following type:

$$(1)\qquad \begin{aligned} P(X_{n+1} = j + 1 \mid X_n = j) &= p_j = \tfrac{1}{2}[1 + \lambda/(j + \lambda)],\\ P(X_{n+1} = j - 1 \mid X_n = j) &= q_j = \tfrac{1}{2}[1 - \lambda/(j + \lambda)],\\ & \qquad j = 0, 1, 2, \cdots, \end{aligned}$$

where λ is a real number greater than $-\frac{1}{2}$. Thus $p_0 = +1$, i.e. the origin is a reflecting barrier. Using the theory of orthogonal polynomials Karlin and McGregor were able to obtain integral representations for the n-step transition probabilities. In particular they showed that

$$(2)\quad P(X_{n+m} = j \mid X_m = i) = \int_{-1}^{1} t^n Q_i(t) Q_j(t)\, d\psi(t) \Big/ \int_{-1}^{1} Q_j^2(t)\, d\psi(t),$$

where $d\psi(t) = c(1 - t^2)^{\lambda - \frac{1}{2}}$ and the Q_j's are the polynomials, orthogonal on $[-1, 1]$ with respect to the weight function $d\psi$, and c is a normalizing constant.

For random walks of the type (1), Lamperti (see [4]) showed that the limiting distribution of the random variables $X_n/n^{\frac{1}{2}}$ exists and moreover he gave an explicit formula for the limiting distribution. For example when $\lambda = \frac{1}{2}$, his result states that

$$(3)\qquad \lim_{n\to\infty} P(X_n/n^{\frac{1}{2}} \leqq t) = \int_0^t s e^{-s^2/2} ds = \Phi(t).$$

Lamperti proved (3) by using the method of moments. It is the purpose of this note to prove (3) in another way; by a method which will yield a stronger result. Our method is to exploit the integral representation (2) in order to obtain a local limit theorem for X_n. This in turn will enable us to conclude the following more delicate asymptotic formula:

$$(4)\qquad P(X_n/n^{\frac{1}{2}} \geqq t_n) \sim 1 - \Phi(t_n)$$

where t_n satisfies the following growth condition:

$$(5)\qquad \lim_{n\to\infty} (t_n^2/n^{\frac{1}{2}}) = 0.$$

A further consequence of (4) is a kind of law of the iterated logarithm for

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