DIMENSIONAL PROPERTIES OF A RANDOM DISTRIBUTION FUNCTION ON THE SQUARE

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The notion of a random distribution function on the line associated with a probability measure μ on the unit square S was introduced and investigated by Dubins and Freedman in [3] and [4]. Certain of its properties were discussed by us in [5]. We now introduce a random distribution function F_ω whose values are probability distributions on the square which has properties expressible in formulae from information theory, as was the case in [5].

The construction of F_ω will be given formally below but it may be helpful to describe it briefly first. F_ω is normalized by setting $F_\omega(S) = 1$. A point (x_ω, y_ω) in S is chosen according to the distribution μ. F_ω is now defined on the dyadic rectangles

$$S_{00} = [(x, y) \mid 0 \leqq x < \tfrac{1}{2}, 0 \leqq y < \tfrac{1}{2}],$$

$$S_{01} = [(x, y) \mid 0 \leqq x < \tfrac{1}{2}, \tfrac{1}{2} \leqq y \leqq 1],$$

$$S_{10} = [(x, y) \mid \tfrac{1}{2} \leqq x \leqq 1, 0 \leqq y < \tfrac{1}{2}]$$

and

$$S_{11} = [(x, y) \mid \tfrac{1}{2} \leqq x \leqq 1, \tfrac{1}{2} \leqq y \leqq 1]$$

by

$$F_\omega(S_{00}) = x_\omega y_\omega$$

$$F_\omega(S_{01}) = x_\omega(1 - y_\omega)$$

$$F_\omega(S_{10}) = (1 - x_\omega)y_\omega$$

$$F_\omega(S_{11}) = (1 - x_\omega)(1 - y_\omega).$$

At the next step the measure in each of the four rectangles is partitioned among its dyadic subrectangles in the same way but independently of each other and of the previous choice (x_ω, y_ω). This process continues and in the limit defines F_ω. Our main result, Theorem 2, gives the dimensions of the supports of F_ω and its margins in terms of various 'average entropies' of μ.

We let $x(n, j) = j/2^n$, $I(n, j) = [x(n, j), x(n, j + 1))$, and $S(n, j, k) = I(n, j) \times I(n, k)$. We also let $I(n, x)$ be that $I(n, j)$ containing x and $S(n, x, y)$ be that $S(n, j, k)$ containing (x, y).

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