INVARIANT PROBABILITIES FOR CERTAIN MARKOV PROCESSES

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1. Introduction. This paper, though self-contained, is concerned with a class of Markov operators closely related to those studied in (Bush and Mosteller, 1953), (Harris, 1952), and (Karlin, 1953).

Throughout this paper, Ω is a set, Γ is a set of mappings of Ω into itself, and P is a probability on Γ. Each P determines a Markov process with Ω for state space and this transition mechanism: when at $\omega \varepsilon \Omega$, choose a $\gamma \varepsilon \Gamma$ according to P, and move to the new state $\gamma(\omega)$. If ω itself is random with distribution μ, then the distribution of the new state is $P\mu$. If $P\mu = \mu$, then μ is *P-invariant.*

Here are some sample results when Ω is compact metric, and Γ is a finite set of uniformly strict, one-to-one contractions of Ω. There is one and only one invariant probability, μ. If P assigns positive mass to each $\gamma \epsilon \Gamma$, then: μ is continuous unless there is a common fixed point for all the $\gamma \varepsilon \Gamma$, in which case μ plainly assigns probability 1 to that point; and the support of μ (that is, the smallest closed set of μ-probability 1) is all of Ω if and only if each point of Ω is in the range of some $\gamma \varepsilon \Gamma$. If m is a probability on Ω, and for all $\gamma \varepsilon \Gamma$, $m\gamma^{-1} \ll m$ (that is, the distribution of γ under m is absolutely continuous with respect to m), then μ is either absolutely continuous or purely singular with respect to m.

In Section 6, Ω is the closed unit interval, and Γ is the set of all linear functions. For this special case, the results are rather complete, and are summarized at the beginning of the section. Some applications are in (Dubins and Freedman, 1966).

The assertions in each section presuppose the hypotheses on Ω, Γ, and P given at the beginning of that section.

2. Each $\gamma \varepsilon \Gamma$ is measurable. Let Ω be a set, $\mathfrak{F}$ be a σ-field of subsets of Ω, and N be the set of all non-negative, finite measures on $\mathfrak{F}$. If N_0 and N_1 are subsets of N, then $N_0 + N_1$ is the set of all $\mu_0 + \mu_1$ for $\mu_0 \varepsilon N_0$ and $\mu_1 \varepsilon N_1$. If $\mu_i \varepsilon N_i$, $\nu_i \varepsilon N_i$, and $\mu_0 + \mu_1 = \nu_0 + \nu_1$ imply $\mu_0 = \nu_0$, then the notation $N_0 \oplus N_1$ is used instead of $N_0 + N_1$. If μ, $\nu \varepsilon N$, then $\mu \leqq \nu$ means $\mu(A) \leqq \nu(A)$ for all $A \varepsilon \mathfrak{F}$. If, for $\sigma \varepsilon N$, $\sigma \leqq \mu$ and $\sigma \leqq \nu$ imply $\sigma = 0$, then $\mu \perp \nu$ (that is, μ is singular with respect to ν). For $M \subset N$, $M^{\perp}$ is the set of all $\nu \varepsilon N$ with $\nu \perp \mu$ for all $\mu \varepsilon M$.

(2.1) Lemma. *$N = M \oplus M^{\perp}$ if and only if $M = M^{\perp\perp}$.*

Proof. Easy. □

(2.2) Lemma. *Let: $M = M^{\perp\perp}$; T be a mapping of N into itself, with $TM \subset M$,*

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