NOTES

A SAMPLE FUNCTION PROPERTY OF MARTINGALES[1]

By D. G. Austin

Northwestern University

We show that if x_n is an L_1 bounded martingale then $\sum_n (x_n - x_{n-1})^2$ converges almost everywhere. We refer to Doob's book [1], Chapter VII, for background material.

THEOREM. *Let* $\cdots x_{-1}, x_0, x_1, \cdots$ *be a martingale, defined on some probability space, such that* $\sup_n \int |x_n| = K < \infty$. *Then* $\sum_{n=-\infty}^{\infty} (x_n - x_{n-1})^2 < \infty$ *almost everywhere.*

PROOF. Let $E_n = \bigcap_{k \leqq n} [w; |x_k(w)| < M]$, $E = \bigcap_n E_n$, where M is a positive real number. Define $\hat{x}_n$ by

$$\begin{aligned} \hat{x}_n(w) &= x_n(w) \quad \text{if} \quad |x_n(w)| < M, \\ &= 0 \quad\quad\;\; \text{if} \quad |x_n(w)| \geqq M. \end{aligned}$$

Then

$$\int_{E_{n-1}} x_n x_{n-1} = \int_{E_{n-1}} x_{n-1}^2 = \int_{E_{n-1}} \hat{x}_{n-1}^2,$$

implying that

$$\int_{E_{n-1}} (\hat{x}_n - \hat{x}_{n-1})^2 = \int_{E_{n-1}} (\hat{x}_n^2 - \hat{x}_{n-1}^2) + 2 \int_{E_{n-1}} (x_n x_{n-1} - \hat{x}_n \hat{x}_{n-1}).$$

Let N be a positive integer. Then

$$\begin{aligned} \sum_{|n| \leqq N} \int_{E_{n-1}} (\hat{x}_n^2 - \hat{x}_{n-1}^2) &= \int_{E_{n-1}} \hat{x}_N^2 + \sum_{-N \leqq n < N} \int_{E_{n-1} - E_n} \hat{x}_n^2 \\ &\quad - \int_{E_{-N-1}} \hat{x}_{-N-1}^2 \\ &\leqq M^2 \end{aligned}$$

Using $x_{n-1} = \hat{x}_{n-1}$ on E_{n-1}, $|x_n - \hat{x}_n| \leqq |x_n|$, and the fact that $\cdots |x_{-1}|, |x_0|, \cdots$ is a submartingale, we have that

$$\begin{aligned} \int_{E_{n-1}} (x_n x_{n-1} - \hat{x}_n \hat{x}_{n-1}) &\leqq M \int_{E_{n-1}} |x_n - \hat{x}_n| \\ &= M \int_{E_{n-1} - E_n} |x_n - \hat{x}_n| \\ &\leqq M \int_{E_{n-1} - E_n} |x_n| \\ &\leqq M \int_{E_{n-1} - E_n} |x_N|, \end{aligned}$$

for all $n \leqq N$, implying that

Received 18 February 1966.

[1] This work was supported by the National Science Foundation.