INDUCTIVE METHODS FOR BALANCED INCOMPLETE BLOCK DESIGNS

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1. Introduction. When working with balanced incomplete block designs, one notes with disappointment that inductive methods are usually unavailable, and in those instances where induction is available it is complicated by the fact that the parameter of induction usually increases by some integral constant greater than unity. It is our purpose here to embed such designs in larger systems where inductive techniques are possible. These larger systems permit block sizes to vary but retain the other axioms of block designs. In this paper we examine two such general systems which are closely related; one is more suitable as a generalization of finite geometries and the other more convenient for use with block designs. The idea of using variable block size is not new. For example it has been used in references [1] and [2]. However it does not seem to have been exploited fully as an inductive technique. We shall introduce a notational system which is favourable to inductive constructions.

2. Balanced incomplete block designs. A balanced incomplete block design is a system consisting of a set V of v objects called varieties, and a collection of b subsets of V called blocks satisfying the following conditions:

B_1 : every block contains precisely $k < v$ (distinct) varieties,

B_2 : every variety occurs in precisely r blocks,

B_3 : every pair of varieties occurs in precisely $\lambda > 0$ blocks.

We shall see directly that B_2 is a redundant axiom. To do so let us introduce the notion of a λ-system.

3. λ-systems. Let us define a λ-system as consisting of a set V of v varieties and a collection of b subsets (called blocks) of V which satisfies the following axioms:

L_1 : every pair of varieties occurs in precisely λ blocks.

L_2 : every block contains at least 2 varieties.

Note that, for $\lambda = 1$, we have the fact that every pair of varieties (points) determines a unique block (line); thus the connection with geometries. We have adopted the terminology of block designs to make the extension of λ-systems to (r, λ)-systems (to be discussed later) more natural.

Associated with every λ-system L there is a sequence of non-negative integers $B = (b_1, b_2, b_3, \cdots)$ where b_i is the number of blocks containing exactly i varieties. Although B is formally an infinite vector, all but a finite number of entries are zero. We shall call such a vector the B-vector of L. Let us note that $b_1 = 0$, and that counting occurrences of pairs of varieties gives

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