A REPRESENTATION FOR CONDITIONAL EXPECTATIONS GIVEN σ-LATTICES[1]

By Tim Robertson

University of Missouri and University of Iowa

1. Introduction. Conditional expectations are discussed by Brunk in [4] and [5]. (Generally, in the literature "conditional expectation" refers to "conditional expectation given a σ-field." However since we shall only be concerned with conditional expectations given σ-lattices, we shall use this abbreviated terminology for the latter, more general, concept.) As illustrated in these references, conditional expectations have been found to provide solutions for several maximum likelihood estimation problems. The principle result of this paper gives a representation for conditional expectations. Marshall and Proschan [7] and the author [9] have found such a representation for estimates useful in studying their asymptotic properties. Special cases of this representation theorem appear in other papers: Theorem 2.2 in Ayer, Brunk, Reid and Silverman [1] and Theorem 1 in Brunk [2] are instances in which the domain of the functions is finite. Brunk, Ewing and Utz [3] consider another version which we shall discuss in detail before proving the theorem.

Suppose we are given a totally finite measure space $(\Omega, \mathfrak{A}, \mu)$ and a σ-lattice $\mathfrak{L}$ of measurable subsets of Ω. A σ-lattice, by definition, contains both the null set $\emptyset$ and Ω and is closed under countable unions and intersections. The symbol $\mathfrak{L}^c$ will denote the σ-lattice of all subsets of Ω which are complements of members of $\mathfrak{L}$. We say that a random variable X is $\mathfrak{L}$-measurable provided $[X > a] \varepsilon \mathfrak{L}$ for each real number a. Let L_2 denote the class of square integrable random variables and $L_2(\mathfrak{L})$ the collection of all those members of L_2 which are $\mathfrak{L}$-measurable. Let $\mathfrak{B}$ denote the class of Borel subsets of the real line. We adopt the following definition for the conditional expectation, $E(X \mid \mathfrak{L})$, of X given $\mathfrak{L}$ (see Brunk [4]).

Definition. If $X \varepsilon L_2$ then $Y \varepsilon L_2(\mathfrak{L})$ is equal to $E(X \mid \mathfrak{L})$ if and only if Y has both of the following properties:

$$(1) \qquad \int (X - Y)Z \, d\mu \leqq 0 \quad \text{for each} \quad Z \varepsilon L_2(\mathfrak{L})$$

and

$$(2) \qquad \int_B (X - Y) \, d\mu = 0 \quad \text{for each} \quad B \varepsilon Y^{-1}(\mathfrak{B}).$$

(Brunk [4] shows that there is such a random variable Y associated with each $X \varepsilon L_2$ and that it is unique in the sense that if W is any other member of $L_2(\mathfrak{L})$ having these properties, then $\mu[Y \neq W] = 0$.)

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