A LIMIT THEOREM FOR MULTIDIMENSIONAL GALTON-WATSON PROCESSES[1]

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1. Introduction. In this paper we consider a positively regular, nonsingular, vector-valued Galton-Watson process. Specifically we consider a temporally homogeneous, k-vector-valued Markov chain, $\{Z_n\,;\, n = 0, 1, \cdots\}$, with among others the following properties:

1. Z_0 is taken to be one of the vectors,

$$e_i = (\delta_{i,1}, \cdots, \delta_{i,k}), \qquad 1 \leqq i \leqq k;$$

2. if P denotes the probability measure of the process, if $Z_n = (Z_n^1, \cdots, Z_n^k)$, $n = 0, 1, \cdots$, and if for each n, $F_{i,j}(x) = P\{Z_{n+1}^j \leqq x \mid Z_n = e_i\}$, $1 \leqq i, j \leqq k$; $x \geqq 0$, then Z_n^j, $1 \leqq j \leqq k$, $0 \leqq n < \infty$, takes on only non-negative integer values and

$$P\{Z_{n+1}^j \leqq x \mid Z_0, \cdots, Z_n\} = F_{1,j}^{Z_n^1} * F_{2,j}^{Z_n^2} * \cdots * F_{k,j}^{Z_n^k}(x),$$

where the right hand side is the convolution of Z_n^i times $F_{i,j}$ for $i = 1, \cdots, k$;

3. if E denotes the expectation functional, if $m_{i,j} = E\{Z_1^j \mid Z_0 = e_i\}$, $1 \leqq i, j \leqq k$, and if M denotes the matrix $(m_{i,j})$, then

$$m_{i,j} = \int_0^\infty x\, dF_{i,j}(x) < \infty, \qquad 1 \leqq i, j \leqq k; \tag{1.1}$$

and there exists a finite positive integer t such that

$$(M^t)_{i,j} > 0, \qquad 1 \leqq i, j \leqq k; \tag{1.2}$$

4. if ρ denotes the largest positive characteristic root associated with M, then

$$\rho > 1. \tag{1.3}$$

We will prove a limit theorem for these processes that we state succinctly below. In the statement of this theorem u and v will be positive right and left eigenvectors of M corresponding to ρ, normalized such that their inner product is 1. (For the existence and properties of ρ, u, and v see our comments below and for a more detailed description of Galton-Watson processes see Chapter II of [3]).

Theorem. *There exists a random vector W and a onedimensional random variable w such that*

$$\lim_{n\to\infty} (Z_n/\rho^n) = W \quad \textit{with probability } 1, \tag{1.4}$$

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