ON THE ASYMPTOTIC EFFICIENCY OF A SEQUENTIAL PROCEDURE FOR ESTIMATING THE MEAN[1]

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1. Introduction. Let the independent, identically distributed random variables

(1) $$X_1, X_2, \cdots$$

be $N(\mu, \sigma^2)$ with μ unknown and $0 < \sigma < \infty$. Define for $n \geqq 2$,

(2) $$\bar{X}_n = n^{-1} \sum_{i=1}^n X_i, \qquad S_n^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2,$$

and suppose that for fixed $s, t > 0$ the loss incurred in estimating μ by $\bar{X}_n$ from a sample of fixed size n is

(3) $$L_n = A|\bar{X}_n - \mu|^s + n^t \qquad (A > 0),$$

with risk

(4) $$\nu_n(\sigma) = E_\sigma L_n = AE_\sigma|\bar{X}_n - \mu|^s + n^t.$$

When σ is *known* the problem of finding the value of n, say n^0, for which the risk (4) is a minimum is perfectly straightforward; let $\nu(\sigma)$,

(5) $$\nu(\sigma) = \nu_{n^0}(\sigma) = \min_{n>0} \nu_n(\sigma),$$

denote the minimum risk. On the other hand, *in ignorance of* σ no procedure based on a fixed number n of observations of (1) will minimize (4) simultaneously for all $0 < \sigma < \infty$. Accordingly, the possibility of utilizing a sample of random size N determined by a certain sequential rule $\mathfrak{R}$ to be specified later, will be considered. In analogy with (3) the loss using $\mathfrak{R}$ is for fixed $s, t > 0$ and N,

(6) $$L_N = A|\bar{X}_N - \mu|^s + N^t \qquad (A > 0),$$

with risk

(7) $$\bar{\nu}(\sigma) = E_\sigma L_N = AE_\sigma|\bar{X}_N - \mu|^s + E_\sigma N^t.$$

It would seem to be of considerable practical importance to compare the values of $\nu(\sigma)$ and $\bar{\nu}(\sigma)$ for values $0 < \sigma < \infty$ of the parameter upon which these functions depend. For, either it will turn out that ν and $\bar{\nu}$ do not differ appreciably for any value of σ, in which case a very useful and easily applied statistical procedure will have been justified, or in the contrary case a horrible example of the dangers of "optional stopping" will have been exposed.

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