A CHARACTERISTIC PROPERTY OF THE MULTIVARIATE NORMAL DISTRIBUTION

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1. Introduction. Suppose that X and Y are two independent $(n \times 1)$ random vectors and the conditional distribution of X given $X + Y$ is known to be multivariate normal. What can we conclude about the distributions of X and Y? The case when X and Y are scalar random variables has been treated in [2].

If X and Y are two independent $(n \times 1)$ random vectors having multivariate normal distributions $N(0, A)$ and $N(0, B)$ respectively (that is with 0 means and symmetric positive definite covariance matrices A, B respectively), it can be easily shown that the conditional distribution of X given $(X + Y)$ is $N[A(A + B)^{-1}(X + Y), \{I - A(A + B)^{-1}\}A]$. Denoting by V the matrix $(I - C)A$ where $C = A(A + B)^{-1}$, it is easy to see that (a) both $V^{-1}C$ and $V^{-1} - V^{-1}C$ are symmetric positive definite and (b) the eigenvalues of C are in $(0, 1)$. These properties are used in establishing the characterization theorem.

2. A characterization theorem. In this section we prove the following theorem.

Theorem. *Let X and Y be two $(n \times 1)$ independent random vectors with continuous probability density functions $f(x)$ and $g(y)$, which are non-vanishing at $x = 0$ and $y = 0$ (0 being the null vector). Let V be a $(n \times n)$ symmetric positive definite matrix and C a non-singular $(n \times n)$ matrix satisfying one of the following two conditions:*

(i) *both $V^{-1}C$ and $V^{-1} - V^{-1}C$ are positive definite, and $V^{-1}C$ is symmetric;*

(ii) *$V^{-1}C$ is symmetric and the eigenvalues of C lie in $(0, 1)$. If the conditional distribution of X given $(X + Y)$ is multivariate normal with mean $C(X + Y)$ and covariance matrix V, then both X and Y are multivariate normal.*

Proof. Denoting the density of $(X + Y)$ by $h(x + y)$, we can write

$$f(x)g(y) = kh(x + y) \exp -\tfrac{1}{2}\{x - C(x + y)\}'V^{-1}\{x - C(x + y)\} \tag{1}$$

where $k = 1/[(2\pi)^{\frac{1}{2}}]^n|V|^{\frac{1}{2}}$.

If in (1) we successively let $x = 0$, $y = 0$, we obtain

$$f(0)g(y) = kh(y) \exp -\tfrac{1}{2}\{Cy\}'V^{-1}\{Cy\}, \tag{2}$$

$$f(x)g(0) = kh(x) \exp -\tfrac{1}{2}\{(I - C)x\}'V^{-1}\{(I - C)x\}. \tag{3}$$

From (1), (2) and (3) after some simplification we have

$$h(x + y) = k'h(x)h(y) \exp -\{(I - C)x\}'V^{-1}\{Cy\}, \tag{4}$$

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