CHARACTERIZATION OF GEOMETRIC AND EXPONENTIAL DISTRIBUTIONS

By Gordon B. Crawford

Boeing Scientific Research Laboratories, Seattle, Washington

1. Introduction. Consider the following property of two independent random variables X and Y:

$$W = \min(X, Y) \quad \text{independent of} \quad X - Y.$$

In [2] Thomas S. Ferguson proves that if X or Y have a discrete part, then this property implies that for suitable constants a, b, $b(X - a)$ and $b(Y - a)$ have (possibly different) geometric distributions; i.e.,:

$$\begin{aligned} P[b(X-a) = n] &= (1-p)p^n, \qquad n \geqq 0, \\ &= 0 \qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

In [3], by the same author, it is shown that if X and Y are absolutely continuous, then for suitable a, $(X - a)$ and $(Y - a)$ have possibly different exponential distributions, i.e.:

$$\begin{aligned} P(X - a \geqq c) &= e^{-c/\lambda}, \qquad c \geqq 0, \\ &= 1 \qquad\qquad \text{otherwise.} \end{aligned}$$

In [1] A. P. Basu gets the same results as [3] under slightly different conditions. It is assumed that X and Y are identically distributed with absolutely continuous distribution $F(\cdot)$; $F(0) = 0$; and the seemingly weaker independence condition:

W, the first order statistic, is independent of the difference $|X - Y|$ of the order statistics.

Basu's result may be obtained from a paper [4] by G. S. Rogers by taking the logarithms of the random variables considered in [4]. Rogers' paper is interesting in that the proof requires only that the regression of $e^{-|X-Y|}$ on W is constant.

In the concluding remarks of [3] Ferguson points out the unsettled problem that arises if X or Y has a singular part. We intend to resolve this problem, assuming that W is independent of $X - Y$.

The main result here is that if the independent random variables X and Y have the property that W is independent of $X - Y$, then X and Y are both geometric random variables or they are both exponential random variables.

We attempt to avoid some measure-theoretic difficulties by working with the distribution functions instead of the densities. The method is different from those mentioned above; all of the results of Ferguson are achieved at little additional expense. Lemmas 1 and 2 are consequences of the asserted independence of W and $X - Y$. Theorem 1 gives a condition which is equivalent to discreteness,

Received 9 August 1965; revised 25 May 1966.