QUASI-LINEARLY INVARIANT PREDICTION

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1. Introduction, the geometry of the problem. Consider a sequence of random variables $X_1, \cdots, X_n, X_{n+1}, \cdots, X_{m+n}$ from which we have observed the first n and we want to predict the value of some known function of the next m random variables.

Let us suppose that the distribution of the random sequence is known except for parameters of location and dispersion (λ, δ) $(-\infty < \lambda < +\infty, 0 < \delta < +\infty)$; this is a restricted but important case, which, in some applications, may be very useful. We will also suppose that the known function $\varphi(X_{n+1}, \cdots, X_{n+m})$ of the next m random variables about which we want to predict is *quasi-linearly invariant*, that is,

$$\varphi(\lambda + \delta X_{n+1}, \cdots, \lambda + \delta X_{n+m}) = \lambda + \delta\varphi(X_{n+1}, \cdots, X_{n+m}).$$

Examples of such functions are the order statistics, and linear combinations of the order statistics.

Let us suppose that the process has a density and the function φ is continuous. Denote by $(1/\delta^{n+1})\mathfrak{L}((x_1 - \lambda)/\delta, \cdots, (x_n - \lambda)/\delta; (z - \lambda)/\delta)$ the likelihood of the random vector $(X_1, \cdots, X_n; Z)$, where $Z = \varphi(X_{n+1}, \cdots, X_{n+m})$. Our purpose is to obtain mean-square predictors and prediction regions for Z, which will be quasi-linearly invariant.

Let R^{n+1} denote $(n + 1)$-dimensional real space. As Z must be quasi-linearly invariant the points $(x_1, \cdots, x_n; z)$ and $(\lambda + \delta x_1, \cdots, \lambda + \delta x_n; \lambda + \delta z)$ are in correspondence by the quasi-linear group of transformations $x \to \lambda + \delta x$ acting in R^{n+1}.

The equivalence relation introduced by this group splits R^{n+1} into equivalence classes which are the half-planes of a bundle, whose axis is the line $x_1 = \cdots = x_n = z$. Each of the half-planes can be described by a system of quantities

$$\xi_3 = (x_3 - x_1)/(x_2 - x_1), \cdots, \xi_n = (x_n - x_1)/(x_2 - x_1), \quad \zeta = (z - x_1)/(x_2 - x_1)$$

invariant under the quasi-linear group acting on R^{n+1}.

2. Best predictors. We now consider the problem of finding a best quasi-linearly invariant predictor, $p(x_1, \cdots, x_n)$, where the loss function is given by $(z - p(x_1, \cdots, x_n))^2$.

If $p(x_1, \cdots, x_n)$ is an invariant predictor, write $h(\xi_3, \cdots, \xi_n) = p(0, 1, \xi_3, \cdots, \xi_n)$. Also, define $\mu(\xi_3, \cdots, \xi_n)$ by

$$\mu(\xi_3, \cdots, \xi_n) = \int_{-\infty}^{+\infty} d\zeta\zeta\mathfrak{L}^*(\xi_3, \cdots, \xi_n; \zeta)/\int_{-\infty}^{+\infty} d\zeta\mathfrak{L}^*(\xi_3, \cdots, \xi_n; \zeta)$$

where $\mathfrak{L}^*(\xi_3, \cdots, \xi_n; \zeta) = \int_0^{+\infty} db b^{n+1} \int_{-\infty}^{+\infty} da \mathfrak{L}(a, a + b, a + b\xi_3, \cdots, a + b\xi_n;$

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