INEQUALITIES FOR LINEAR COMBINATIONS OF ORDER STATISTICS FROM RESTRICTED FAMILIES

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1. Introduction. In this paper we present some results of theoretical interest concerning order statistics and their spacings from certain restricted families of positive random variables. Applications to life testing are discussed in a separate paper [Barlow and Proschan (in process)].

For a specified continuous distribution G for which $G(0) = 0$, we consider the family $\mathfrak{F}$ of distributions such that for F in $\mathfrak{F}$ and $F(0) = 0$, $G^{-1}F$ is starshaped or convex on the support of F. Distributions related in this way by convexity have been studied by Van Zwet (1964). It is known that $F(0) = 0$, $G(0) = 0$, and $G^{-1}F$ convex imply $G^{-1}F$ starshaped. [Bruckner and Ostrow (1962).]

If G is the exponential distribution, then $G^{-1}F$ convex where finite is equivalent to F having an increasing failure rate (i.e., F is IFR). $G^{-1}F$ starshaped is equiv alent to F having an increasing failure rate average (i.e., F is IFRA) [Birnbaum, Esary, and Marshall (1965)]. $G^{-1}F$ concave on $[0, \infty)$ is equivalent to F having decreasing failure rate (i.e., F is DFR).

If G is the uniform distribution, then $G^{-1}F$ convex on the support of F is equivalent to F having an increasing density. If $F(G)$ denotes the gamma distribution with shape parameter $\alpha(\beta)$ with $\alpha > \beta$, then $G^{-1}F$ is convex on $[0, \infty)$ [Van Zwet (1964)]. The Weibull family is similarly ordered, as may be readily verified.

Comparisons for linear combinations of expected values of order statistics from F and G are obtained when $G^{-1}F$ is starshaped. In addition, stochastic comparisons for linear combinations of order statistics are obtained when $G^{-1}F$ is convex as well as when $G^{-1}F$ is starshaped.

Specializing to the case where G is the exponential distribution and F is IFR or IFRA, stochastic comparisons are made for the "total time on test," which is of interest in life testing. Bounds on the expected values of order statistics are also obtained for this case.

Finally, we investigate the preservation of certain class properties under the operation of taking order statistics.

2. Preliminaries. Throughout this paper we adopt the following notation and assumptions. Let $X(Y)$ have distribution $F(G)$. We assume that $F(0) = 0 = G(0)$, and that F and G are continuous. We also assume that the support of F is

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