ON A FACTOR AUTOMORPHISM OF A NORMAL DYNAMICAL SYSTEM

By D. Newton and W. Parry

University of Sussex

0. Introduction. In this paper we exhibit a factor of a normal dynamical system which, although possessing a similar spectral structure, is not in general isomorphic to a normal dynamical system. In Section 1 we compute the entropy and canonical system of measures associated with the factor decomposition. In Section 2 we obtain a spectral decomposition for the factor automorphism which resembles very closely the usual spectral decomposition of a normal dynamical system. The results of Section 2 are used in Section 3 to give an example of a dynamical system with countable Lebesgue spectrum, and zero entropy which is mixing of all orders. Such an example according to Rohlin [8], has been found by Girsanov (unpublished).

For the theory of Lebesgue spaces and the associated concepts of measurable partitions, homomorphisms, unitary rings c.f. [7], [9].

A dynamical system $(X, \mathcal{B}, m, T)$ (abbreviated to (X, T)) is a Lebesgue space $(X, \mathcal{B}, m)$ together with an automorphism T of $(X, \mathcal{B}, m)$.

Let U_T be the unitary operator induced by T defined on $L^2(X)$ by

$$U_T f = fT,$$

then U_T is an automorphism of the unitary ring $L^2(X)$. If L is a unitary sub-ring of $L^2(X)$ such that $U_T L = L$ we refer to (L, U_T) as a unitary subsystem of the unitary system $(L^2(X), U_T)$.

$(X', \mathcal{B}', m', T')$ is said to be a factor of $(X, \mathcal{B}, m, T)$ if there is a homomorphism ϕ of $(X, \mathcal{B}, m)$ onto $(X', \mathcal{B}', m')$ such that $\phi T = T'\phi$. In this case U_ϕ defined by

$$U_\phi f' = f'\phi$$

will be a ring isomorphism of $(L^2(X'), U_{T'})$ into a unitary subsystem of $(L^2(X), U_T)$.

Let (X', T') be a factor of (X, T) under the homomorphism ϕ. Let $X_{x'} = \{x : \phi(x) = x'\}$ and $\mathcal{B}_{x'} = \{B \cap X_{x'} : B \varepsilon \mathcal{B}\}$; then for almost all $x' \varepsilon X'$ there exists a normalised measure $m_{x'}$ such that $(X_{x'}, \mathcal{B}_{x'}, m_{x'})$ is a Lebesgue space and for every $B \varepsilon \mathcal{B}$, $m_{x'}(B)$ is a measurable function such that

$$\int_{X'} m_{x'}(B)\, dm' = m(B).$$

The measures $m_{x'}$ are called the canonical system associated with the factor decomposition $\phi^{-1}(x')$ and are unique $[m']$.

One way of defining the canonical system $m_{x'}$ is by the formula

$$\int_{X_{x'}} f_n(y)\, dm_{x'} = E(f_n \mid \phi^{-1}\mathcal{B}'),$$

Received 21 March 1966.