ON THE EXPECTED VALUE OF A STOPPED MARTINGALE[1]

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Throughout this note, $X_1, X_2, \cdots$ is a martingale, and $K = \sup_n E|X_n|$. As is easily verified, $E|X_t| \leqq K$ for every stopping time t. This note studies the existence of t such that $E|X_t| = K$ when $K = \infty$, and finds necessary and sufficient conditions on the distribution of the martingale for $E(X_t)$ to be equal to $E(X_1)$ for all t.

Theorem 1. *If* $\sup_n E|X_n| = \infty$, *then there is a stopping time* t *for* $X_1, X_2, \cdots$ *such that* $E|X_t| = \infty$.

Let $\mathfrak{F}_j$ be the σ-field generated by $X_1, \cdots, X_j$. As usual, a *stopping time* t for $X_1, X_2, \cdots$ is a random variable whose range is the set of positive integers with $+\infty$ adjoined, such that for each n, the event $\{t = n\} \varepsilon \mathfrak{F}_n$. Say t is *finite* if it is finite almost surely. Whether or not t is finite, $E|X_t|$ is evaluated as $\int_{t<\infty}|X_t|$.

Of course, $E|X_t|$ may be finite for all finite stopping times t, and yet be infinite for some stopping time t. Here is an example which helped us find Theorems 1 and 2. Let $X_1 = 0$. On $X_n \neq 0$, let $X_{n+1} = X_n$ a.e. On $X_n = 0$, given $X_1, \cdots, X_n$, let $X_{n+1} = 0$ with conditional probability $1 - 2p_{n+1}$, while $X_{n+1} = x_{n+1}$ and $X_{n+1} = -x_{n+1}$ with conditional probability p_{n+1} each. Let $0 < p_n < \frac{1}{2}$, $\sum p_n < \infty$, $0 < x_n < \infty$, and $\sum p_n x_n = \infty$.

Let

$$V_j = \sup_{n \geq j} E\{|X_n| \mid \mathfrak{F}_j\}. \tag{1}$$

Lemma 1. *With the understanding that the* V'*s may be infinite on a set of positive measure,* $V_1, V_2, \cdots$ *is a martingale relative to* $\mathfrak{F}_1, \mathfrak{F}_2, \cdots$.

Proof. Plainly, V_j is $\mathfrak{F}_j$-measurable and

$$\begin{aligned} E\{V_{j+1} \mid \mathfrak{F}_j\} &= E\{\lim_n E[|X_n| \mid \mathfrak{F}_{j+1}] \mid \mathfrak{F}_j\} \\ &= \lim_n E\{E[|X_n| \mid \mathfrak{F}_{j+1}] \mid \mathfrak{F}_j\} \\ &= \lim_n E\{|X_n| \mid \mathfrak{F}_j\} \\ &= V_j. \end{aligned}$$

Lemma 2. *If* t *is a stopping time for an integrable stochastic process* $Y_1, Y_2, \cdots$, $\mathfrak{F}$ *is a* σ-*field of measurable sets,* n *is a positive integer, and the event* $\{t = n\} \varepsilon \mathfrak{F}$, *then almost everywhere on* $\{t = n\}$,

$$E\{Y_t \mid \mathfrak{F}\} = E\{Y_n \mid \mathfrak{F}\}. \tag{2}$$

Proof. Since both sides of (2) are plainly $\mathfrak{F}$-measurable, it is only necessary

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