MARTINGALE TRANSFORMS[1]

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1. Introduction. Let $f = (f_1, f_2, \cdots)$ be a martingale on a probability space $(\Omega, \mathcal{A}, P)$. Let $d_1 = f_1, d_2 = f_2 - f_1, \cdots$ so that $f_n = \sum_{k=1}^n d_k$, $n \geqq 1$. It is convenient to say that $g = (g_1, g_2, \cdots)$ is a *transform* of f if $g_n = \sum_{k=1}^n v_k d_k$, where v_n is a real $\mathcal{A}_{n-1}$-measurable function, $n \geqq 1$, and $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$ are σ-fields such that $\{f_n, \mathcal{A}_n, n \geqq 1\}$ is a martingale. Note that g need not be a martingale. It is easy to see that g is a martingale if and only if $E|g_n|$ is finite for all n. This condition is satisfied, for example, if each v_n is bounded. Transforms of real (but not of extended real) submartingales may be defined similarly.

Such transforms, particularly in the case in which the v_n may take only 0 and 1 as possible values, have a long history and an interesting gambling interpretation. See Halmos [5], Doob [3], and some of the earlier work referred to in [5]. The emphasis is sometimes not on the transform g itself but on related sequences $\{g_{m_n}, n \geqq 1\}$ where the m_n are stopping times. Halmos's skipping theorem and Doob's optional stopping and sampling theorems, which give conditions assuring that $\{g_{m_n}\}$ is a martingale or a submartingale, are examples (g may equal f).

We prove here that, under mild conditions, martingale transforms converge almost everywhere. We also prove several related almost everywhere convergence theorems for martingales and establish a number of inequalities that follow from this convergence.

Inequalities and almost everywhere convergence results for the sequences $\{g_{m_n}\}$ mentioned above, whether or not they are martingales or submartingales, follow immediately.

Our first result (Theorem 1) is that a transform g of an L_1 bounded martingale f converges almost everywhere on the set where the maximal function v^* of the multiplier sequence $v = (v_1, v_2, \cdots)$ is finite. Here $v^*(\omega) = \sup_n |v_n(\omega)|$, the boundedness condition on f is that $\sup_n E|f_n| < \infty$, and g converging almost everywhere means that $\lim_{n\to\infty} g_n(\omega)$ exists and is finite for almost all ω.

Consider a gambler faced with the prospect of winning d_n dollars playing game n in an infinite sequence of games. Under $\sup_n E|f_n| < \infty$ and the usual condition of fairness (the expectation of d_n given $\mathcal{A}_{n-1}$, his experience before playing game n, is 0), his sequence f of fortunes $f_n = \sum_{k=1}^n d_k$ converges almost surely to a finite limit f_∞. Is there anything that he can do to make his fate more interesting? Suppose that, under new rules, he can win $v_n d_n$ rather than d_n dollars, where he is allowed to choose v_n just before the nth game on the basis of his past experience $\mathcal{A}_{n-1}$. Can he choose $v = (v_1, v_2, \cdots)$ subject to $v^* < \infty$ so that his new

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