SOME CONVERGENCE THEOREMS FOR INDEPENDENT RANDOM VARIABLES[1]

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1. Introduction. Let a_{nk}, a_n be real numbers and x_n be independent random variables. The convergence of $\sum_{k=1}^{\infty} a_{nk}x_k$ as $n \to \infty$ has been discussed in [4], [5], [7], [14] and [15]. Section 2 of this paper is suggested by Hill's work [7], and is devoted to the convergence of $\sum_{k=1}^{\infty} a_{nk}x_k$ under the condition $\sum_{k=1}^{\infty} a_{nk}^2 = o(\log^{-1} n)$. As an application, we prove the following theorem, relating to a result of Pruitt [14]. *If x_n are identically distributed, $Ex_n = 0$, $Ex_n^2 = 1$, and $\sum_{k=1}^{n} a_{nk}^2 = 1$, then $n^{-\frac{1}{2}}\sum_{k=1}^{n} a_{nk}x_k$ tends to zero* a.e. Section 3 is suggested by Kahane's work [9]. Salem-Zygmund's sample continuity theorem [15] for $\sum_1^{\infty} a_n x_n \cos nt$ is extended from Bernoulli random variables to generalized Gaussian random variables (defined in Section 2). Sections 4 and 5 are devoted to the extensions of Hsu-Robbins' complete convergence theorem [8]; the material in these two sections is closely related to the work of Franck and Hanson [4].

The first counter-example showing that a directed set indexed martingale of bounded variation may diverge pointwise is due to Dieudonné [1]. A simpler counter example is given in Section 6. Section 7 contains some theorems about a.e. unconditional convergence of sums of independent identically distributed random variables, and in Section 8 the following theorem is proved. *If $E \sup_n |x_n| < \infty$, then $\sum_1^{\infty} x_n$ converges* a.e. *implies that $\sum_1^{\infty} Ex_n$ converges.*

2. Extension of Hill's theorems. In this section, we assume that for $n, k = 1, 2, \cdots$, a_{nk} are real numbers and $A_n = \sum_{k=1}^{\infty} a_{nk}^2 < \infty$ for each n.

LEMMA 1. *Let x be a random variable, $Ex = 0$ and $|x| \leqq 1$. Then for every real number t,*

$$Ee^{tx} \leqq e^{t^2}. \tag{1}$$

PROOF. If $0 < t \leqq 1$, then $E \exp [tx] \leqq 1 + t^2 \leqq \exp [t^2]$. If $t > 1$, then $E \exp [tx] \leqq \exp [t] \leqq \exp [t^2]$. By symmetry, we obtain (1).

In [9], a symmetric random variable x is said to be semi-Gaussian, if there exists $\alpha \geqq 0$ such that for every real number t

$$E \exp [tx] \leqq \exp [\alpha^2 t^2/2]. \tag{2}$$

The minimum of those α satisfying (2) is denoted by $\tau(x)$. Obviously, a $N(0, 1)$ random variable x is semi-Gaussian (with $\tau(x) = 1$), and by Lemma 1, if x is symmetric and bounded by K, x is semi-Gaussian (with $\tau(x) \leqq 2^{\frac{1}{2}}K$). For

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