ADDITIONAL LIMIT THEOREMS FOR INDECOMPOSABLE MULTI-DIMENSIONAL GALTON-WATSON PROCESSES[1]

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1. Introduction. In this paper we consider an indecomposable, non-singular, vector-valued Galton-Watson process. Specifically we consider a temporally homogeneous, k-vector-valued Markov chain, $\{Z_n ; n = 0, 1, \cdots\}$, with among others the following properties, assumed throughout this paper.

(1) Z_0 is taken to be one of the vectors,

$$e_i = (\delta_{i,1}, \cdots, \delta_{i,k}), \qquad 1 \leqq i \leqq k;$$

(2) If P denotes the probability measure of the process, if $Z_n = (Z_n^1, \cdots, Z_n^k)$, $n = 0, 1, \cdots$, and if for each n

$$F_{i,j}(x) = P\{Z_{n+1}^j \leqq x \mid Z_n = e_i\}, \qquad 1 \leqq i, j \leqq k; x \geqq 0,$$

then Z_n^j, $1 \leqq j \leqq k$, $0 \leqq n < \infty$, takes on only non-negative integer values and

$$P\{Z_{n+1}^j \leqq x \mid Z_0, \cdots, Z_n\} = F_{1,j}^{Z_n^1} * F_{2,j}^{Z_n^2} * \cdots * F_{k,j}^{Z_n^k}(x),$$

where the right hand side is the convolution of Z_n^i times $F_{i,j}$ for $i = 1, \cdots, k$;

(3) If E denotes the expectation functional, if $m_{i,j} = E\{Z_1^j \mid Z_0 = e_i\}$, $1 \leqq i, j \leqq k$, and if M denotes the matrix, $(m_{i,j})$, then

$$m_{i,j} = \int_0^\infty x\, dF_{i,j}(x) < \infty, \qquad 1 \leqq i, j \leqq k, \tag{1.1}$$

and for each pair, i, j, there exists an integer $t = t(i, j) \geqq 1$ such that

$$(M^{t(i,j)})_{i,j} > 0. \tag{1.2}$$

(4) If ρ denotes the largest positive characteristic root associated of M, then

$$\rho > 1. \tag{1.3}$$

We call a branching process satisfying (1.2) *indecomposable.* Whenever the integer t in (1.2) is independent of the pair i, j, then both M and the Z-process are called positively regular. We will extend the results obtained in [4] to processes that are indecomposable but not positively regular. We will also for indecomposable processes present several limit theorems of a type that has received little attention so far (see, however, the acknowledgement at the end of the paper). In a forthcoming paper [5] we shall show how to extend many of the results obtained here to the case of decomposable Galton-Watson processes, i.e. to processes that do not satisfy (1.2), but otherwise satisfy conditions (1)–(4).

Since M is non-negative and finite, it follows from the Perron-Frobenius

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