NOTES

ON MEASURES EQUIVALENT TO WIENER MEASURE[1]

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Recently Shepp [5] has given, among several other results, a simple condition for the equivalence of a Gaussian measure to Wiener measure. His proof is ingenious but long. The purpose of this note is to record a somewhat simpler proof of this result which we had obtained independently. Our proof uses a reproducing kernel Hilbert space (*rkhs*) criterion, due to Oodaira [2], for the equivalence of Gaussian measures.

We first state Shepp's theorem (in a terminology more familiar to engineers):

Discrimination between a zero-mean Gaussian process with covariance function $R(t, s)$ and a Wiener process with covariance function $\min(t, s)$, $[0 \leqq t, s \leqq T]$ will be nonsingular[2] if and only if there exists a unique, symmetric, square-integrable function on $T \times T$,

$$K(t, s) = K(s, t), \qquad \int_0^T \int_0^T K^2(t, s)\, dt\, ds < \infty \tag{1}$$

such that

$$R(t, s) - \min(t, s) = \int_0^t \int_0^s K(u, v)\, du\, dv \tag{2}$$

and

$$1 \text{ is not an eigenvalue of } K(u, v) \tag{3}$$

We shall, as mentioned above, obtain this result by direct application of the following theorem[3] of Oodaira [2]: *Discrimination between zero-mean Gaussian processes with covariance functions* $R_i(t, s)$, $t, s \varepsilon T \times T$, $i = 1, 2$, *will be nonsingular if and only if* (i) $R_1 - R_2$ *belongs to the direct product space* $H(R_2) \otimes H(R_2)$, *where* $H(R_2)$ *is the reproducing kernel Hilbert space* (*rkhs*) *of* R_2 *and* (ii) *there exist positive* (>0) *constants* c_1 *and* c_2 *such that* $c_1R_2 \leqq R_1 \leqq c_2R_2$.

We shall apply this theorem when $R_1(t, s) = R(t, s)$ and $R_2(t, s) = \min(t, s)$, $0 \leqq t, s \leqq T$. We need to calculate $H(R_2) \otimes H(R_2)$. This is easy to do and ac-

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[2] That is, the measures corresponding to these two processes will be equivalent.

[3] We refer for definitions to Oodaira [2] or Parzen [4]. A similar theorem was given by Capon [1]. Both these theorems are extensions of, and were suggested by, the work of Parzen [3] [4]. We note that other rkhs proofs are possible, see, e.g., J. I. Golosov, Soviet Math. Doklady 7 (1966), 48–51. This reference was also noted by Shepp [5].