FUNCTIONS OF FINITE MARKOV CHAINS[1]

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1. Statement of the Problem. This paper is a presentation of some results in the study of random processes which arise as functions of finite, continuous parameter Markov chains. For reference purposes throughout this paper the definition of such processes is contained in:

Hypothesis H: The processes $X(t)$ and $Y(t)$ will be said to satisfy hypothesis H whenever $X(t)$ is a basic Markov chain (Definition 2.1) with state space $\mathfrak{N} = \{1, 2, \cdots, N\}$, there is a function f mapping $\mathfrak{N}$ onto $\mathfrak{M} = \{1, 2, \cdots, M\}$, where $M \leqq N$, and the process $Y(t)$ is equal in joint distribution to the process $f[X(t)]$.

The process $Y(t)$ is termed a function of a finite Markov chain. The process $Y(t)$ need not be Markov and in fact the question motivating this research is that of finding necessary and sufficient conditions for $Y(t)$ to be Markov. Such conditions are given in Theorem 2.5. The conditions given are in terms of exponential type processes (Definition 2.2) which arise as functions of basic Markov chains (Theorem 2.3), and their order (Definition 2.4).

Aspects of this problem have been considered previously by C. J. Burke and M. Rosenblatt [1], M. Rosenblatt [11], J. Hachigian and M. Rosenblatt [8] and J. Hachigian [7]. Burke and Rosenblatt [1] gave necessary and sufficient conditions for $Y(t)$ to be Markov when $X(t)$ was a discrete parameter reversible Markov chain. They also gave necessary and sufficient conditions for $Y(t)$ to be Markov whatever the initial probabilities in the case when $X(t)$ was a continuous time parameter Markov chain. J. Hachigian and M. Rosenblatt [8] extended the results of [1] to reversible, continuous time parameter Markov processes with arbitrary state space. Related questions concerning functions of discrete time finite Markov chains have been considered by Gilbert [6], Dharmadhikari [3], [4], [5] and most recently by Heller [9] who completed the problem of characterizing processes which arise as functions of Markov chains.

This paper is organized in four main sections. Section 2 is introductory and contains a statement of the main theorem. The theorem of Section 3 has interest of its own, in that it identifies regeneration states (Definition 3.1) for certain exponential type processes. The results in Section 4 are special ones needed for the proof of Theorem 2.5 which makes up Section 5.

2. Basic Markov chains and exponential type processes. Let $X(t)$ be a standard [2, p. 123] Markov chain with a stationary transition matrix $P(t) = (P_{ij}(t))$, and initial column vector $\mathbf{P} = (P_i)$ whose state space is

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