ASYMPTOTIC DISTRIBUTION OF LINEAR COMBINATIONS OF FUNCTIONS OF ORDER STATISTICS WITH APPLICATIONS TO ESTIMATION[1]

By Herman Chernoff, Joseph L. Gastwirth and M. V. Johns, Jr.

Stanford University

1. Introduction. The research leading to the results reported in this paper was originally motivated by a desire to find asymptotically optimal linear combinations of order statistics for estimating location and scale parameters in both the uncensored and censored data cases. It was recognized at the outset that the methods of Chernoff-Savage [2] could be applied to obtain the asymptotic normal distribution of statistics of the form

$$T_n = n^{-1} \sum J[j/(n+1)]X_{jn},$$

where $X_{1n} \leqq X_{2n} \leqq \cdots \leqq X_{nn}$ are the ordered observations of a sample, and where $J(\cdot)$ is a well behaved function. It was also discovered that a simple, standard variational argument could be used to obtain the asymptotically optimal J's explicitly for the case of estimating location and scale parameters.

Subsequently, the attention of the authors was directed to the unpublished dissertation of Carl Bennett [1] where the asymptotically optimal J's had been obtained for both the uncensored and multicensored cases by a tour de force which did not include a derivation of the asymptotic normality of the estimates. Some of Bennett's results were obtained independently by Jung [5] under rather restrictive conditions. Plackett [9], and Weiss [13], independently considered the case where all observations below the pth and above the qth sample percentiles $(0 < p < q < 1)$ are censored, and obtained asymptotic normality for suitable linear combinations of the available order statistics. Plackett also characterized the asymptotically optimal weights for this case. Asymptotic normality for the case of uncensored data was not treated by these authors.

The present authors found that, while the Chernoff-Savage approach was adequate for the particular applications initially considered, it nevertheless involved certain objectionable mathematical inelegancies. A different technique based on representing the ordered observations in terms of independent exponentially distributed random variables was therefore selected. This technique yields stronger results and yet involves only arguments which are essentially elementary. Regrettably, these results still seem to fall short of the "best possible" results which may ultimately require a sophisticated "invariance principle" type of argument.

Received 8 November 1965.

[1] This research of these authors was sponsored in part by the Office of Naval Research, Contracts Nonr-225(52)(NR-342-022) and Nonr-225(53)(NR-042-002). Reproduction in whole or in part is permitted for any purpose of the United States Government.