LIPSCHITZ BEHAVIOR AND INTEGRABILITY OF CHARACTERISTIC FUNCTIONS[1]

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1. I shall establish some theorems connecting the asymptotic behavior of a distribution function F and the local behavior of its characteristic function φ.

Theorem 1. *If* $0 < \gamma < 1$, *we have* $\varphi \varepsilon \operatorname{Lip} \gamma$ *if and only if*

$$F(x) - F(\pm \infty) = O(|x|^{-\gamma}), \qquad |x| \to \infty. \tag{1.1}$$

Condition (1.1) is to be read as $F(x) = O(|x|^{-\gamma})$ as $x \to -\infty$ and $1 - F(x) = O(x^{-\gamma})$ as $x \to +\infty$.

More precisely, $\varphi(x) - \varphi(0) = O(|x|^{\gamma})$, or even $\varphi(x) + \varphi(-x) - 2\varphi(0) = O(|x|^{\gamma})$, implies (1.1); $\varphi \varepsilon \operatorname{Lip} \gamma$ is implied by (1.1). Hence if a characteristic function satisfies a Lipschitz condition of order γ, $0 < \gamma < 1$, at the origin, then it satisfies a Lipschitz condition of the same order at all points.

Theorem 1 fails for $\gamma = 1$. The problem of finding something similar for $\gamma = 1$ is of special interest because it is connected with the problem of the existence of the derivative of a characteristic function at the origin. Let Λ^* and λ^* be the classes of continuous functions φ such that $\varphi(x + h) + \varphi(x - h) - 2\varphi(x) = O(h)$ or $o(h)$, uniformly in x, as $h \to 0$ (λ^* is the class of smooth functions); Λ or λ at x means the same thing for this particular x.

Theorem 2. *We have* $\varphi \varepsilon \Lambda^*$ *or* λ^* *if and only if*

$$F(x) - F(\pm \infty) = O(1/|x|) \quad \text{or} \quad o(1/|x|), \qquad |x| \to \infty. \tag{1.2}$$

More precisely, $\varphi \varepsilon \Lambda$ *or* λ *at* 0 *implies* (1.2); $\varphi \varepsilon \Lambda^*$ *or* λ^* *is implied by* (1.2). *Hence in particular* $\varphi \varepsilon \lambda^*$ *if and only if* φ *is smooth at* 0.

Zygmund [3] showed that $\varphi'(0)$ exists if and only if φ is smooth at 0 and

$$\lim_{T \to \infty} \int_{-T}^{T} t \, dF(t) \quad \text{exists.} \tag{1.3}$$

Pitman [2] showed that $\varphi'(0)$ exists if and only if $F(x) - F(\pm \infty) = o(1/|x|)$ and (1.3) holds. By Theorem 2, we have φ smooth (either at 0 or everywhere) if and only if $F(x) - F(\pm \infty) = o(1/|x|)$, so that the corresponding parts of Zygmund's and Pitman's conditions are really equivalent. In Section 4 I give a short deduction of Pitman's theorem from Theorem 2. (For another proof see Feller [1], p. 528.)

If φ is smooth we can also show that $\varphi'(x)$ exists (for a particular x) if and only if

$$\lim_{T \to \infty} \int_{-T}^{T} te^{ixt} \, dF(t)$$

exists.

Theorems 1 and 2 say that $\varphi(x + h) \to \varphi(x)$ with a specified rapidity if and

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