A MEAN-SQUARE-ERROR CHARACTERIZATION OF BINOMIAL-TYPE DISTRIBUTIONS

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1. Hodges and Lehmann (1950), with an acknowledgement of priority to Rubin, have shown that, if θ^* is an estimator of the parameter θ based on a single observation on the binomial distribution

$$\text{prob}\,\{X = x\} = \binom{N}{x}\theta^x(1-\theta)^{N-x}, \qquad \text{for } x = 0, 1, \cdots, N, \tag{1}$$

with N fixed and known and a loss function $W(\theta, \theta^*) = (\theta - \theta^*)^2$, the minimax estimator is

$$\theta^M = (X + \tfrac{1}{2}N^{\frac{1}{2}})/(N + N^{\frac{1}{2}}). \tag{2}$$

This is also the Bayesian estimator when θ has a Bayesian prior probability element of the form

$$d\lambda(\theta) = (\theta(1-\theta))^{\frac{1}{2}N^{\frac{1}{2}}-1}\,d\theta/B(\tfrac{1}{2}N^{\frac{1}{2}}, \tfrac{1}{2}N^{\frac{1}{2}}). \tag{3}$$

This distribution of θ is therefore a least favourable one for the estimator θ^M. The proofs are simplified by the facts that

(i) θ^M is linear in the random variable X,

(ii) the mean square error (mse) or risk function of θ^M is independent of θ.

2. Since the binomial family of distributions is one in which the sample mean of a constant number of independent observations is both a sufficient statistic and a minimum variance unbiased estimator (mvue) of their common expectation, it is of interest to investigate whether the two properties, (i) and (ii), apply to any other families of distributions which possess minimum variance unbiased estimators.

One may, therefore, seek to replace the binomial by other "exponential-type" distributions. In the notation of Lehmann's book (1959), page 50, (1) might be replaced by the probability element

$$C(\theta)e^{Q(\theta)T(x)}h(x)\,d\mu(x), \tag{4}$$

in which μ is a measure function; for then the sample mean of $T(X)$ is the mvue of $ET(X)$. It is clear, however, that little is gained in the present problem by using (4) instead of the simpler formulation,

$$e^{\alpha x - F(\alpha)}\,d\mu(x). \tag{5}$$

This formulation, or, rather, a closely related one, was found convenient in a paper (1947) in which I called it a Laplacian distribution, and it has also (cf.

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