A TECHNICAL LEMMA FOR MONOTONE LIKELIHOOD RATIO FAMILIES

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Let $\mathcal{A}$ be a σ-algebra on an abstract set X. Let $\mathcal{P}/\mathcal{A}$ be an ordered family of p-measures which is dominated by a σ-finite measure $\mu/\mathcal{A}$. Let p be a density of $P \varepsilon \mathcal{P}$ with respect to μ. $\mathcal{P}$ is said to have monotone likelihood ratios, if there exists an $\mathcal{A}$-measurable map $T: X \to R$, defined $\mathcal{P}$-a.e., such that for each pair P', $P'' \varepsilon \mathcal{P}$ with $P' < P''$ there exists a nondecreasing function $H_{P'P''}: R \to \bar{R}$ such that

$$(1) \qquad p''(x)/p'(x) = H_{P'P''}(T(x)) \qquad \mu\text{-a.e.},$$

whenever $p''(x)/p'(x)$ is defined.

In this formulation, the μ-null set on which (1) is violated depends on P', P''. The purpose of this note is to show that for MLR-families the densities can always be chosen such that (1) holds except on a fixed μ-null set. This leads to a simplification in many proofs connected with MLR-families, (e.g. those connected with the theory of uniformly most powerful tests).

Lemma. *If $\mathcal{P}$ is a* MLR-*family there exists a dominating σ-finite measure μ_0 and a coherent system of densities with respect to μ_0, such that for each pair P', $P'' \varepsilon \mathcal{P}$ with $P' < P''$ the ratio $p''(x)/p'(x)$ is a nondecreasing function of $T(x)$ with the exception of a fixed μ_0-null set. In other words: There exists a subset $X_0 \subset X$ such that $P(X_0) = 1$ for all $P \varepsilon \mathcal{P}$ and such that $p''(x)/p'(x)$ is a nondecreasing function of $T(x)$ for all $x \varepsilon X_0$ for which this ratio is defined.*

Proof. As $\mathcal{P}/\mathcal{A}$ is dominated by a σ-finite measure $\mu/\mathcal{A}$, according to the lemma of Halmos and Savage (see Lehmann, p. 354, Theorem 2), there exists a σ-finite measure $\mu_0 = \sum_{n=1} c_n \cdot P_n$, $P_n \varepsilon \mathcal{P}_0$, which is equivalent to $\mathcal{P}$. The densities of $P/\mathcal{A}$ with respect to $\mu_0/\mathcal{A}$ can be assumed to depend on x only through $T(x)$ (Lehmann, p. 48, Theorem 8). Hence for the following proof we might change the p-space: instead of $(X, \mathcal{A}, \mathcal{P})$ we will consider $(\mathcal{R}, \mathcal{B}, \mathcal{Q})$, where $\mathcal{B}$ is the Borel-algebra on $\mathcal{R}$ and $\mathcal{Q}/\mathcal{B}$ is the family of p-measures induced by $\mathcal{P}/\mathcal{A}$ through T (i. e., $Q(B) = P(T^{-1}B)$). Let $\nu_0/\mathcal{B}$ be the p-measure induced by $\mu_0/\mathcal{A}$. Then $\nu_0/\mathcal{B}$ is equivalent to $\mathcal{Q}/\mathcal{B}$. If $q(t)$ is a density of $Q/\mathcal{B}$ with respect to $\nu_0/\mathcal{B}$, $p(x) := q(T(x))$ is a density of $P/\mathcal{A}$ with respect to $\mu_0/\mathcal{A}$.

As $\mathcal{B}$ is separable, $\mathcal{Q}/\mathcal{B}$ is separable with respect to uniform convergence, i. e., there exists a countable subset $\mathcal{Q}_0 \subset \mathcal{Q}$ such that for each $Q \varepsilon \mathcal{Q}$ there exists a sequence $(Q_n)_{n=1,2,\cdots}$ with $Q_n \varepsilon \mathcal{Q}_0$ such that $Q_n(B) \to Q(B)$ uniformly in $B \varepsilon \mathcal{B}$. (See Lehmann, p. 352, Theorem 1.)

For each $Q \varepsilon \mathcal{Q}_0$, we fix a version, say q, of $dQ/d\nu_0$. For convenience we choose

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