ON THE EXPECTED VALUE OF A STOPPED SUBMARTINGALE

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Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a submartingale in a probability space $(\Omega, \mathfrak{F}, P)$. A stopping time t is a positive integer-valued ($+\infty$ included) random variable such that for each $n = 1, 2, \cdots$, the set $[t = n] \, \varepsilon \, \mathfrak{F}_n$. If a stopping time t is finite, i.e., $P[t = \infty] = 0$, then t is said to be a stopping rule. For a stopping time t, $E|x_t|$ is defined to be $\int_{[t<\infty]} |x_t| \, dP = \int_{[t<\infty]} |x_t|$. In a recent paper of Dubins and Freedman [2], the following theorem has been proved. Let $(x_n, \mathfrak{F}_n, n > 1)$ be a martingale. (a) If $\sup E|x_n| = \infty$, then there exists a stopping time t such that $E|x_t| = \infty$. (b) If $\sup E|x_n| < \infty$, then $Ex_t = Ex_1$ for every stopping rule t if and only if x_n's are uniformly integrable. However, the proof is somewhat complicated. In an oral communication, D. Siegmund has given a simpler proof of (b), by using the standard martingale arguments. In this note, Dubins and Freedman's results (a) and (b) are extended to submartingales as follows:

THEOREM. *Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a submartingale* (a). *If* $\sup E(x_n^-) = \infty$, *then there exists a stopping time t such that* $E|x_t| = \infty$. (b) If $\sup E|x_n| < \infty$,

$$Ex_m \leqq Ex_t \leqq \sup Ex_n \tag{1}$$

for every stopping rule t satisfying $P[t \geqq m] = 1$, if and only if x_n's are uniformly integrable.

PROOF. (a) We can assume that $E|x_n| < \infty$ for each n and $Ex_1 = 0$. Put $C_0 = \Omega$. Then there exists an integer $n_1 > 1$ such that $\int_{C_0} x_{n_1}^- \geqq 1$. Put $A_1 = [x_{n_1} \leqq 0]C_0$ and $B_1 = C_0 - A_1$. Then either $\sup \int_{A_1} x_n^- = \infty$ or $\sup \int_{B_1} x_n^- = \infty$. Hence we can choose $C_1 = A_1$ or B_1 so that $\sup \int_{C_1} x_n^- = \infty$. Put $D_1 = A_1$ if $C_1 = B_1$ and $D_1 = B_1$ if $C_1 = A_1$. When $D_1 = A_1$, by the definition of n_1,

$$\int_{D_1} |x_{n_1}| = \int_{A_1} |x_{n_1}| = \int_{C_0} x_{n_1}^- \geqq 1,$$

and when $D_1 = B_1$, by submartingale property and $Ex_1 = 0$,

$$\int_{D_1} |x_{n_1}| = \int_{B_1} |x_{n_1}| = \int_{C_0} x_{n_1}^+ \geqq 1.$$

Assume that we have defined n_k, A_k, B_k, C_k and D_k for a positive integer k. Choose n_{k+1}, A_{k+1}, B_{k+1}, C_{k+1} and D_{k+1} as follows. Since $\sup \int_{C_k} x_n^- = \infty$, there exists an integer $n_{k+1} > n_k$ such that $\int_{C_k} x_{n_{k+1}}^- \geqq 1$. Put $A_{k+1} = [x_{n_{k+1}} \leqq 0]C_k$ and $B_{k+1} = C_k - A_{k+1}$. Then either $\sup \int_{A_{k+1}} x_n^- = \infty$ or $\sup \int_{B_{k+1}} x_n^- = \infty$. Hence we can choose $C_{k+1} = A_{k+1}$ or B_{k+1} so that $\sup \int_{C_{k+1}} x_n^- = \infty$. Put $D_{k+1} = A_{k+1}$ if $C_{k+1} = B_{k+1}$ and $D_{k+1} = B_{k+1}$ if $C_{k+1} = A_{k+1}$. When $D_{k+1} = A_{k+1}$, by the definition of n_{k+1},

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