ON STATIONARY MARKOV PROCESSES[1]

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1. Introduction. Consider Markov processes $(X_n, n \geqq 0)$ with given stationary transition probabilities and (σ-finite) stationary measure α. The state space Ω is arbitrary; Σ is a σ-field of measurable subsets of Ω. First, we prove that the strictly stationary process $(X_n, n \geqq 0)$ is embeddable in a strictly stationary Markov process $(X_n, -\infty < n < \infty)$ which we call the *extended process* (see [5]). This was a fact assumed true in [5], but no proof was given. We also examine the invariant random variables for these processes in Theorem 2. Also briefly discussed is the reversed Markov process. In the event that Ω is the real or complex field, Theorem 1 is known ([1], p. 456) and if α is finite Theorem 2 is known ([1], pp. 458–460). However, counterexamples are offered illustrating the difficulties arising when α is infinite.

This note is a sequel to [5]. Besides the gap there mentioned above, the language of [5] suggested that Theorem 2 is true in general, i.e., without condition (A). Section 4 of this note will set matters straight.

2. Main results.

Lemma. *Let Σ be separable, that is, Σ is generated by a countable family of sets. Then the strictly stationary Markov process $(X_n, n \geqq 0)$ may be embedded in an extended process $(X_n, -\infty < n < \infty)$.*

Proof. Consider bilateral sequence space Ω_1 with elements $\omega = (\cdots \omega_{-1}, \omega_0, \omega_1, \cdots)$. Let Λ_0 and ${}_0\Lambda$ be the σ-fields generated by cylinders in Ω_1 with non-negative coordinates and non-positive coordinates respectively. Using the transition probabilities, for each x a conditional probability measure $P(\cdot \mid X_0 = x)$ may be constructed on Λ_0 according to [1], p. 614. With α as initial measure on X_0-space, it is easily seen that a shift-invariant measure α_0 may be defined on Λ_0 by putting $\alpha_0(U) = \int P(U \mid X_0 = x)\alpha(dx)$ for $U \varepsilon \Lambda_0$ (see Lemma 1 of [5]). Proceed as in [1], p. 456, to assign a mass α_1 to cylinder sets in Ω_1 by setting $\alpha_1(C) = \alpha_0(T^{-j}C)$ where $T^{-j}C \varepsilon \Lambda_0$, T is the shift, and C is a cylinder of Ω_1. To prove that α_1 determines a measure on the σ-field Σ_1 of Ω_1 determined by the cylinder sets (and hence that $(X_n, n \geqq 0)$ is embedded in $(X_n, -\infty < n < \infty)$) it is necessary to prove α_1 countably additive on the cylinders.

Kolmogorov's extension theorem fails because Ω here is arbitrary. It is already known that α_1 restricted to Λ_0 is countably additive and equal to α_0. Now we check α_1 restricted to ${}_0\Lambda$ is countably additive. To see this, observe that since $X_0, X_1, \cdots$ is a Markov process with initial distribution α, the process $\cdots X_n, X_{n-1}, \cdots, X_0$ is also Markovian (see [1], p. 83; the restriction to real

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