ON THE LACK OF A UNIFORMLY CONSISTENT SEQUENCE OF ESTIMATORS OF A DENSITY FUNCTION IN CERTAIN CASES

By R. H. Farrell

Cornell University[1]

1. Introduction. We will let C_α be the set of distribution functions F on $R = (-\infty, \infty)$ with the following properties:
(1) F has a first derivative f_F defined and continuous at all points of R, such that $\sup_{x\varepsilon R} f_F(x) \leqq \alpha$.
(2) F has a second derivative defined and continuous at all points of R.

Let $\{\delta_n, n \geqq 1\}$ be a sequence of functions such that if $n \geqq 1$ then δ_N is a real valued Borel measurable function on R_N (Euclidean n-space.) Let $\{X_n, n \geqq 1\}$ be a sequence of independently and identically distributed random variables such that if F is the distribution function of X_1 then $F \varepsilon C_\alpha$. We let N be a stopping variable relative to $\{X_n, n \geqq 1\}$ and F, and consider the sequential estimator $\delta_N = \delta_N(X_1, \cdots, X_N)$ of $f_F(0)$.

Loss will be measured by square error, so that the risk function of $\delta = \{\delta_n, n \geqq 1, X_m, m \geqq 1, N\}$ is

$$R(F, \delta) = \sum_{n=1}^\infty \int \cdots \int_{\{N=n\}} (\delta_n(x_1, \cdots, x_n) - f_F(0))^2 \prod_{i=1}^n f_F(x_i)\, dx_i .$$

Theorem. *Suppose that $\alpha \geqq 3$ is a real number. If $\sup_{F\varepsilon C_\alpha} E_F N < \infty$, then $\sup_{F\varepsilon C_\alpha} R(F, \delta) \geqq \frac{1}{16}$.*

Since fixed sample size procedures satisfy the hypothesis of the theorem, we conclude that $\sup_{F\varepsilon C_\alpha} R(F, \delta) \geqq \frac{1}{16}$ for every choice of a fixed sample size procedure and choice of $\alpha \geqq 3$. It should be observed that this theorem does not deny the existence of a consistent sequence of estimators. For example, if $\{X_n, n \geqq 1\}$ are independently and identically distributed, if the distribution function of X_1 is F, and if F_n is the sample distribution function based on $X_1, \cdots, X_n$, $n \geqq 1$, then let $\beta = \frac{1}{3}$ and define $\delta_n = (F_n(n^{-\beta}) - F_n(-n^{-\beta}))/(2n^{-\beta})$. Then $\lim_{n\to\infty} E_F\delta_n = f_F(0)$ so that asymptotically the bias of δ_n goes to zero. Clearly the variance of δ_n goes to zero. The theorem says in effect that no *uniformly* consistent sequence of estimators exists relative to the class C_α.

In recent years there have appeared several papers discussing methods of estimation of the value of a density function. Papers which have come to our attention are Parzen [4] and Leadbetter [3]. In addition the author has had no difficulty in inventing several methods of estimation quite different from those of Parzen and Leadbetter (it is not our purpose to discuss these here.) All these methods have a common characteristic. If C_α^* is the set of F with continuous

Received 14 July 1966.

[1] Research supported in part by the Office of Naval Research under contract NONR 401(50).