RATES OF CONVERGENCE OF ESTIMATES AND TEST STATISTICS[1]

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0. Introduction. This paper contains brief descriptions of certain large sample theories of estimation and testing null hypotheses. The classical asymptotic variance theory of estimation is considered in Section 1; a parallel and closely related development based on probabilities of large deviations in Section 2; and a relatively unexplored viewpoint involving the rate at which the estimate itself approaches the true value in Section 3. Sections 4–7 describe a version of testing in which any given test statistic is evaluated in terms of the rate at which it makes the null hypothesis more and more incredible as the sample size increases when a non-null distribution obtains.

The statistical framework considered throughout the paper is the following: X is an abstract sample space of points x. The probability distribution of x is determined by an abstract parameter θ which takes values in a set Θ. $s = (x_1, x_2, \cdots, \text{ad inf})$ is a sequence of independent observations on x. For each $n = 1, 2, \cdots, T_n = T_n(s)$ is a real valued statistic which depends on s only through $(x_1, \cdots, x_n)$. Most of the propositions stated formally are versions of propositions in [5], [7], [9], and [10]. Sufficient conditions for the validity of the propositions are discussed in an appendix, and all proofs are deferred to the appendix.

PART I. POINT ESTIMATES

$g(\theta)$ is a real valued parametric function defined on Θ. It is required to estimate the value of g.

1. Asymptotic variance. Suppose that T_n is a consistent and asymptotically normal estimate of g with asymptotic variance v/n, i.e., there exists $v(\theta)$, $0 < v < \infty$, such that for each θ

$$n^{\frac{1}{2}}(T_n(s) - g(\theta))/(v(\theta))^{\frac{1}{2}} \rightarrow \mathfrak{N}(0, 1) \text{ in distribution} \tag{1}$$

as $n \rightarrow \infty$ when θ obtains. For any $\epsilon > 0$ and any θ let

$$\alpha_n(\epsilon, \theta) = P_\theta(|T_n(s) - g(\theta)| \geqq \epsilon). \tag{2}$$

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