THE GENERALIZED VARIANCE: TESTING AND RANKING PROBLEM[1]

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In this note it is shown that, for a sample from a multivariate normal distribution, the density function of the sample generalized variance possesses a monotone likelihood ratio (MLR). This result is used to construct a uniformly most powerful invariant test for a testing problem concerning the population generalized variance. Also, the result is applied to the problem of ranking multivariate normal populations according to the size of their generalized variances.

Let $X_1, \cdots, X_{n+1}$ be a random sample from a $p(p \leqq n)$ variate normal distribution $N_p(\mu, \Sigma)$, with mean μ and nonsingular covariance matrix Σ. Consider the sufficient statistic $(\bar{X}, S)$ where

$$\bar{X} \equiv (1/n+1) \textstyle\sum_{i=1}^{n+1} X_i \tag{1}$$

and

$$S \equiv \textstyle\sum_{i=1}^{n+1} X_i'X_i - (n+1)\bar{X}'\bar{X}, \tag{2}$$

so that $\bar{X}$ and S are independent, $\bar{X}$ is $N_p(\mu, 1/(n+1)\Sigma)$ and S has a Wishart distribution, $W_p(\Sigma, n)$, with expectation $n\Sigma$. If we set $\theta = \det(\Sigma)$ and $V = \det(\mathrm{S})$, then $\theta(V$, resp.) is the population (sample, resp.) generalized variance. It is well known that V has the same distribution as $\theta \prod_{i=1}^{p} \chi^2_{n-i+1}$ where the factors χ^2_{n-i+1} are independent and have a chi-square distribution with $n - i + 1$ degrees of freedom (see Anderson (1958) p. 171). Let $f_p(v, \theta)$ denote density function of V.

LEMMA 1. *The density function, $f_p(v, \theta)$, of the generalized variance has a MLR.*

PROOF. The proof is by induction on $p(1 \leqq p \leqq n)$. For $p = 1$, V has the density of a scaled chi-square random variable which is known to have a MLR. Now, it is straightforward to show that

$$f_p(v, \theta) = \textstyle\int_0^\infty f_{p-1}(v, x)h(x, \theta)\, dx \tag{3}$$

where h is the density of a scaled χ^2_{n-p+1} random variable. Noting that $h(x, \theta)$ has a MLR, the result now follows by the induction hypothesis and an application of a result due to Karlin (1956, Lemma 5, p. 125). ▯

As an application of the above lemma, consider the hypothesis $H_0: \theta \leqq c_1$ and

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