AN INTRINSICALLY DETERMINED MARKOV CHAIN[1]

By J. MacQueen[2]

Stanford University

Consider a Markov chain $X_0, X_1, X_2, \cdots$, on the non-negative integers with $P[X_{t+1} = x - 1 \mid X_t = x] = \gamma(x)$, $P[X_{t+1} = x + 1 \mid X_t = x] = 1 - \gamma(x)$, for $x = 1, 2, \cdots$, and with 0 an absorbing state; that is, $P[X_{t+1} = 0 \mid X_t = 0] = 1$. Thus the process can either go up or down one step and for $x \geqq 1$, $\gamma(x)$ is the probability of going down. For $x = 0, 1, \cdots$, let $q_\gamma(x) = P[X_t = 0$ *for some* $t \geqq 0$ *given* $X_0 = x]$. Hence $q_\gamma(0) = 1$ by definition.

Now let φ be a given function on the interval $(0, 1)$ satisfying $0 \leqq \varphi \leqq 1$. We impose on the function γ of the preceding paragraph the condition

$$\gamma(x) = \varphi(q_\gamma(x)), \qquad x = 1, 2, \cdots. \tag{1}$$

Thus the transition law of the process depends on the probability that the process is absorbed at zero; but the probability that this happens depends on the transition law. The process, if it is determined at all, is determined by its own behavior, i.e., it is determined 'intrinsically'.

We show the possibility of determining a process in this fashion, by virtue of the following result:

Theorem 1. *If φ is uniformly continuous on $(0, 1)$ with $0 < a = \inf \varphi \leqq \sup \varphi = b < \frac{1}{2}$, then there exists a function γ such that γ and q_γ jointly satisfy (1); these functions, γ and q_γ, are unique if φ is non-increasing.*

For the sake of a phenomenological interpretation, we can imagine that X_t is the fortune of a man who works at gambling. The harder he works each day, the greater the probability of his winning one unit, and the less the probability of losing one unit, these being the only outcomes. Being concerned with the probability of becoming destitute, but at the same time not particularly liking to work, he assesses the latter probability each day, and thereby decides how hard to work, generally working less the lower this probability is. Theorem 1 says that choice of such a procedure, as expressed by φ, is theoretically possible and leads to a unique mode of behavior.

Proof of Theorem 1. The proof will make use of the following elementary lemmas.

Lemma 1. *If $\gamma = 1 - \bar{\gamma}$ satisfies $0 < \gamma(x) < 1$, $x = 1, 2, \cdots$, then there is at most one function q satisfying*

Received 15 April 1966; revised 12 January 1967.

[1] This research was supported in part by the Ford Foundation under its grant to the Western Management Science Institute at the University of California, Los Angeles, and in part by the Office of Naval Research (Contract 233(75)).

[2] Visiting Associate Professor of Statistics.