SLOWLY BRANCHING PROCESSES[1]

By Wolfgang J. Bühler

University of California, Berkeley

A (discrete time) branching process is a sequence Z_n, $n = 1, 2, \cdots$, of random variables describing the size of successive generations of a population. Here it is assumed that given $Z_n = k$ the $n + 1$st generation consists of the offspring of the k members of the nth generation; each of these k individuals having a random number U_i $(i = 1, \cdots, k)$ of children. The U_i are mutually independent with a common distribution given by its probability generating function $f(s)$.

A continuous time Markov branching process is usually described as representing the number Z_t of particles at time t. Any one of the particles present at time t has a probability $b\tau + o(\tau)$ of splitting within the time interval $[t, t + \tau]$ into a random number of new particles whose distribution is given by a probability generating function $h(s)$.

Under somewhat less restrictive assumptions on the processes Stratton and Tucker [6] have considered a sequence $\{Z_N(t), N = 1, 2, \cdots\}$ of branching processes with $Z_N(0) = N$ and such that, as $N \to \infty$, the branching rate b_N converges to zero with Nb_N approaching a finite limit b assuming that $h(s)$ is independent of N. They found that the sequence of processes $\{X_N(t)\} = \{Z_N(t) - N\}$ converges to a process with independent increments and with characteristic function

$$\psi(u, t) = \exp \{bte^{-iu}[h(e^{iu}) - e^{iu}]\}. \tag{1}$$

The interpretation of this result is that, as N increases, the branching of the process is slowed down so much that in the limit the occurrence of "higher generation" particles can be neglected and out of the original particles only a certain number (with Poisson distribution as limit of binomials) will have split. Thus the limiting process will be a compound Poisson process (see e.g. Feller [3]).

The purpose of this note is to establish the discrete time version and the continuous state space versions of the above result and at the same time to give a proof simpler than that by Stratton and Tucker.

Let us first consider the discrete time case. Here slowing down the process is achieved by letting $P(Z_1 = 1 \mid Z_0 = 1)$ tend to one without changing the conditional distribution of Z_1 given $Z_1 \neq 1$. In other words we consider a sequence $\{Z_n(N), N = 1, 2, \cdots\}$ of branching processes with $Z_0(N) = N$ such that the distribution of the number of 'offspring' of any one individual is given by

$$f(s, p) = (1 - p)s + pf(s), \qquad |s| \leqq 1, \tag{2}$$

where $f(s)$ may be any probability function and $Np \to \lambda > 0$.

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