THE POWER OF THE LIKELIHOOD RATIO TEST

By Bradley Efron

Stanford University and the RAND Corporation

1. Introduction. Suppose we are given n independent and identically distributed observations $x_1, x_2, \cdots, x_n$ of a random variable X having density function $f(x)$ with respect to some measure $\mu(x)$ on a measurable space Ω, and are asked to test the simple hypothesis $f(x) \equiv f_0(x)$ versus the simple alternative $f(x) \equiv f_1(x)$ at a significance level α, $0 < \alpha < 1$. It is well-known that the most powerful test, which rejects for large values of the likelihood ratio

$$\prod_{i=1}^{n} (f_1(x_i)/f_0(x_i)),$$

has an "error probability of the second kind" (probability of mistakenly accepting the null hypothesis) $\beta_n(\alpha)$ satisfying

$$\lim_{n\to\infty} (\log \beta_n(\alpha)/n) = -I, \tag{1}$$

where I is the Kullback-Leibler information number

$$I = E_0(\log (f_0(X)/f_1(x))) = \int_\Omega (\log (f_0(x)/f_1(x)))f_0(x)\, d\mu(x). \tag{2}$$

A nice proof of (1), which requires no additional assumptions, can be found in Section 4 of [4].

Here it is shown that if we make the additional assumption that

$$E_0(|\log (f_0(X)/f_1(X))|^3) < \infty,$$

(E_0 always indicating expectation under the null hypothesis), a better limiting expression for $\beta_n(\alpha)$ can be derived which is sensitive enough to allow power comparisons between different levels of α. In Section 3 the usefulness of similar expressions for simple numerical approximation of the function $\beta_n(\alpha)$ in small samples is illustrated.

In addition to the information number I defined above, let

$$J = E_0(\log (f_0(X)/f_1(X)) - I)^2 \tag{3}$$

and

$$K = E_0(\log (f_0(X)/f_1(X)) - I)^3, \tag{4}$$

which are both finite by the previous assumption. Then we have the following:

Theorem. *If* $\log (f_0(X)/f_1(X))$ *is not a lattice random variable under the null hypothesis, then*

$$\beta_n(\alpha) = \exp \{-[nI - (nJ)^{\frac{1}{2}}z_\alpha + (K/6J)(1 - z_\alpha^2) + \tfrac{1}{2}z_\alpha^2]\} \cdot (2\pi nJ)^{-\frac{1}{2}}(1 + o_n(1)) \tag{5}$$

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