A NOTE ON STATISTICAL EQUIVALENCE[1]

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1. Introduction. Bahadur [1 p. 292] has, in effect, remarked that Blackwell's use of sufficiency for experiments agrees with the classical definition of a sufficient statistic. More precisely: a statistic $t: X \to Y$ is sufficient for a set $\mathfrak{M}$ of measures of X in the sense of Halmos and Savage [8] if and only if the experiment induced on Y by T is sufficient for $\mathfrak{M}$ in the sense of Blackwell [2], [3], provided certain technical conditions are fulfilled. Bahadur derives the non-trivial half of this statement from the main theorem of [1] (cf. [7], [9]). Here, the analogue of Bahadur's result will be given in a technically different context and some related questions will be discussed. The proofs will be as self-contained as possible and, in particular, they will not depend on the theorem of [1] or any other deep results from other papers, except in Section [8] where a result from [10] is used. The relations between Bahadur's results and ours are discussed in the final section. A paper of DeGroot [6] gives some interesting applications of Blackwell's concept of sufficiency.

2. The main theorem. First a revised version of some definitions from [10] will be given. If Ω is a Borel field ($=\sigma$-algebra $=\sigma$-field) of subsets of a set X, a subcollection $\Omega_0 \subset \Omega$ will be called a *σ-ideal* if Ω_0 is closed under countable unions and if the intersection of an element of Ω with an element of Ω_0 is an element of Ω_0. The example which motivates the concept is: if $\mathfrak{M}$ is a set of probability measures on (X, Ω) the set of elements E of Ω such that $m(E) = 0$ for every m in $\mathfrak{M}$ is a σ-ideal. The notation "mod Ω_0" will be used to indicate that the subset of X on which an assertion fails to hold is an element of Ω_0. A *statistical operation* from a triple (X, Ω, Ω_0) to another triple (Y, Λ, Λ_0) is a real valued function T on a subset of $\Lambda \times X$ which satisfies:

(i) for each F in Λ, $T(F, x)$ is defined mod Ω_0 and is an Ω-measurable function of x satisfying $0 \leqq T(F, x) \leqq 1$ mod Ω_0 and $T(Y, x) = 1$ mod Ω_0.

(ii) if $F_1, F_2, \cdots$ is a sequence of pairwise disjoint elements of Λ, then $T(\bigcup_{i=1}^{\infty} F_i, x) = \sum_{i=1}^{\infty} T(F_i, x)$ mod Ω_0.

(iii) if F is in Λ_0, $T(F, x) = 0$ mod Ω_0. If Ω_0 is empty, T is just a stochastic transformation in the sense of Blackwell [2],]3], or a transition measure in the sense of Čencov [5]. Although T is a map from a subset of $\Lambda \times X$ to the real numbers in the sense of set theory, T is also a map from (X, Ω, Ω_0) to (Y, Λ, Λ_0) in the sense of category theory, hence it will often be written $T: (X, \Omega, \Omega_0) \to (Y, \Lambda, \Lambda_0)$. The composition of two statistical operations $T: (X, \Omega, \Omega_0) \to (Y, \Lambda, \Lambda_0)$ and $S: (Y, \Lambda, \Lambda_1) \to (Z, \Sigma, \Sigma_0)$ can be defined if $\Lambda_1 \supseteqq \Lambda_0$ by a slight

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