STOCHASTIC POINT PROCESSES: LIMIT THEOREMS[1]

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1. Introduction. Stochastic point processes correspond to our intuitive notion of a countable aggregate of points randomly distributed in R^n (Cartesian n-space). For clarity, the points of the aggregate will be called *particles* and so we shall be concerned with random distributions of particles. The point processes on R^1 that have been most widely studied are the Poisson process, renewal processes, processes with stationary increments and general stationary processes. Khintchine [11] proved a variety of general statements about point processes on R^1 (or random streams as he called them). However most of the published results about point processes on R^1 make essential use of the order properties of the line. The first interesting examples of point processes in higher dimensions seem to be the cluster processes in R^3 introduced in [15] by Neyman and Scott as models for the distribution of clusters of galaxies. Here we begin a systematic study of limit theorems for stochastic point processes for all R^n.

Sections 2 and 3 contain the basic definitions and examples and a lemma fundamental to our later theorems. Sections 3–7 consider a variety of operations which "scramble" a point process. Well distributed processes are introduced as the natural class upon which to perform these operations. Our results lead us to the following heuristic principle:

If one scrambles a point process without introducing any new dependence between particles and if the operation is iterated, then the resulting sequence of scrambled processes converges to a mixture of Poisson processes. This reinforces our notion of the Poisson process as the most random distribution of particles.

The results in this paper are true for all R^n, but for the sake of clarity, I present them in R^2. The generalization of our results to all R^n are immediate. *Thus from this point on, unless explicitly stated otherwise, all statements and proofs refer to point processes on R^2. Furthermore all sets in R^2 are assumed to be bounded Borel sets unless otherwise indicated.*

2. Definitions and examples. Let ω be a countable (finite or denumerably infinite) aggregate of particles in R^2 and let $S \subset R^2$. Then $N(S, \omega)$ denotes the number of particles of ω in S. We generally write $N(S)$ in place of $N(S, \omega)$, the set ω being understood.

Definition 2.1. *A stochastic* (*or random*) *point process on* R^2 is a triple (M, M_B, P), where (1) M is the class of all countable aggregates of particles in R^2 without limit points, (2) M_B is the smallest Borel algebra on M such that for

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