ALTERNATIVE PROOFS FOR CERTAIN UPCROSSING INEQUALITIES

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1. Introduction. The purpose of this paper is to give different methods of proof for certain upcrossing and downcrossing inequalities that appear in martingale theory and to obtain sometimes improved inequalities. They include the fundamental one, due to Doob for martingales, and others due to Bishop, Dubins, Hunt and Snell. Except for the last two sections, the paper deals with an inductive approach which is exhibited first to show the main idea in the case of Snell's extension to submartingales of Doob's inequality. For the other cases fewer details will be given. For the sake of completeness we shall repeat some definitions.

Given a finite sequence of points $\mathcal{C} = \{c_1, \cdots, c_n\}$ in the two-point compactification of the real line and two real numbers $a < b$, we say that $\mathcal{C}$ *upcrosses* (*downcrosses*) $[a, b]$ *at least m times* if there exist m pairs of integers: $j_1 < k_1 < \cdots < j_m < k_m$ such that $c_{j_i} \leqq a$, $c_{k_i} \geqq b$, $(c_{j_i} \geqq b, c_{k_i} \leqq a)$. We say that $\mathcal{C}$ *upcrosses m times the interval* $[a, b]$ if $\mathcal{C}$ upcrosses it at least m times but not $m + 1$. A finite sequence of measurable functions in a probability space (Ω, Σ, P), $\{f_1, \cdots, f_n\}$, is said to be a *submartingale* if they are integrable and $\int_{F_j} f_j \, dP \leqq \int_{F_j} f_{j+1} \, dP$, for any j and any $F_j \varepsilon \mathcal{B}(f_1, \cdots, f_j)$, the least σ-algebra making measurable $f_1, \cdots, f_j$. It is said to be a *supermartingale* if their negatives constitute a submartingale, and is called a *martingale* if it is simultaneously a sub and a supermartingale.

We want to prove the following inequality (cf. [1], [6], [2]):

$$E(U) \leqq (b - a)^{-1} E[(f_n - a)^+], \tag{1}$$

where $U = U_{a,b,R}(\omega)$ denotes the number of times the submartingale $R = \{f_1, \cdots, f_n\}$ upcrosses $[a, b]$ at ω. The finite sequence $S = \{(f_1 - a)^+/(b - a), \cdots, (f_n - a)^+/(b - a)\}$ is also a submartingale, and everywhere, we have:

$$U_{0,1,S}(\omega) = U_{a,b,R}(\omega).$$

Therefore, it is sufficient to prove (1) in case of a nonnegative submartingale and $a = 0$, $b = 1$:

$$E(U_{0,1}) \leqq E(f_n). \tag{2}$$

(In general, we shall drop superfluous indices without comment.) When $n = 2$, (2) follows immediately:

$$P(f_1 = 0, f_2 \geqq 1) = E(U) \leqq \int_{\{f_1 = 0\}} f_2 \, dP. \tag{3}$$

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