ON THE COMBINATION OF INDEPENDENT TEST STATISTICS

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1. Introduction. Let T_i be independent one-sided test statistics for testing the hypothesis $H_{i,0}: \theta_i = \theta_{i,0}$ for the independent real-valued parameter θ_i against the one-sided alternatives $\theta_i > \theta_{i,0}$, $i = 1, 2, \cdots, k$. For the sake of definiteness we suppose that large values of T_i lead to rejection of $H_{i,0}$. It is desired to combine the results of these tests, i.e. to construct a function of $T_1, T_2, \cdots, T_k$ that may be used to test the combined hypothesis $H_0: \theta_i = \theta_{i,0}$, $i = 1, 2, \cdots, k$, against the alternative $\theta_i \geqq \theta_{i,0}$, $i = 1, 2, \cdots, k$, with strict inequality at least once.

A well-known combination method is the so-called omnibus test of R. A. Fisher [4] which is based on the probability integral transformation. If T_i has a continuous distribution function F_i under the null-hypothesis $H_{i,0}$, then $F_i(T_i)$ is uniformly distributed on $(0, 1)$ under $H_{i,0}$. As a result, under H_0, $-\log(1 - F_i(T_i))$, $i = 1, 2, \cdots, k$, have independent exponential distributions, hence

$$-\sum_{i=1}^{k} \log(1 - F_i(T_i))$$

has a gamma distribution with parameter k and consequently a chi-square test is applicable. Independent of Fisher's work, K. Pearson [12] proposed $-\sum_{i=1}^{k} \log F_i(T_i)$ as a test statistic, small values leading to rejection of H_0. L. H. C. Tippett [13] considered $\max_{1 \leqq i \leqq k} F_i(T_i)$, whereas B. Wilkinson [15] put forward the mth largest value among the $F_i(T_i)$, which has a beta distribution under H_0. A. Birnbaum [1] has shown, however, that for the exponential class of distributions Pearson's test and Wilkinson's test for $m > 1$ are inadmissible.

Generalizing the approach of Fisher and Pearson, T. Liptak [10] studied statistics of the type $\sum_{i=1}^{k} \alpha_i \Psi^{-1}(F_i(T_i))$, where Ψ^{-1} is the inverse of an arbitrary distribution function Ψ and α_i are arbitrary weights. Taking for Ψ the exponential distribution one obtains a weighted version of Fisher's test which was introduced by I. J. Good [5]. However, from the point of view of distribution theory a more obvious choice is Liptak's proposal to consider $\sum_{i=1}^{k} \alpha_i \Phi^{-1}(F_i(T_i))$, where Φ denotes the standard normal distribution function. Under H_0 this statistic is normally distributed for any set of weights.

H. O. Lancaster [9] suggested another way to add weights to Fisher's test by transforming $1 - F_i(T_i)$ to gamma (or chi-square) distributed variates with possibly different parameter values. He also gave an approximate likelihood-ratio procedure for combining k identical tests against the same simple alternative and discussed asymptotic theory and weighting.

The validity of all tests based on the probability integral transformation de-

Received 17 October 1966.