INFINITE DIVISIBILITY OF INTEGER-VALUED RANDOM VARIABLES

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1. Introduction. Proof of infinite divisibility (i.d. for brevity) of distributions such as the Poisson, the Normal and, the Gamma is available in text books. The proof hinges on the fact that if $\phi(t, \theta)$ is the characteristic function of the given distribution, then $\{\phi(t, \theta)\}^n$ can be written in the form $\phi(t, \theta^*)$. Clearly, one cannot expect this to happen in general situations. For example, the characteristic function $\phi(t, \theta) = \log\{1 - \theta \exp(it)\}/\log\{1 - \theta\}$ of the logarithmic distribution does not permit this form. The aim of this paper is to develop conditions which are necessary and sufficient if an integer valued random variable is to be i.d. and to illustrate their use with examples. The interesting part of these conditions is that they are explicit. Because of this explicitness, numerical methods can be used to give counter examples when a distribution is not i.d. and to gather inspiration to work out algebraic proofs if the numerical methods indicate that the distribution could be i.d.

2. Necessary and sufficient conditions. Let $g(z)$ be the probability generating function (pgf for brevity) of a random variable X taking values in $\{0, 1, 2, \cdots\}$ with probabilities P_i, $P_0 \neq 0$. If X is i.d., then for every integer n, we have $X = \sum_{i=1}^n X_i$ where X_i are independent and identically distributed. Clearly, X_i takes on the values $(0, 1, 2 \cdots)$. Denote the probabilities in the distribution of X_i by π_i and the pgf by $h(z)$. Then, $h(z) = \sum \pi_i z^i$ and

$$g(z) = \{h(z)\}^n. \tag{1}$$

On differentiating (1) with respect z, we get on rearranging the terms,

$$(\textstyle\sum_{i=0}^\infty (i + 1)P_{i+1}^* z^i)(1 + \sum_{i=1}^\infty \pi_i^* z^i/n) = (1 + \sum_{i=1}^\infty P_i^* z^i)(\sum_{i=0}^\infty (i + 1)\pi_{i+1}^* z^i), \tag{2}$$

where $P_i^* = P_i/P_0$ and $\pi_i^* = n\pi_i/\pi_0$. On ignoring the term $\sum_{i=1} \pi_i^* z^i/n$ in comparison with unity on the left-hand side of (2) we get

$$\textstyle\sum_{i=0}^\infty (i + 1)P_{i+1}^* z^i = (1 + \sum_{i=1}^\infty P_i^* z^i)(\sum_{i=0}^\infty (i + 1)\pi_{i+1}^* z^i). \tag{3}$$

On equating coefficients of z^i and solving for π_i^*, we get

$$\pi_i^* = (-1)^{i+1} \begin{vmatrix} P_1^* & 1 & 0 & \cdots & 0 & 0 \\ P_2^* & P_1^* & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ P_{i-1}^* & P_{i-2}^* & P_{i-3}^* & \cdots & P_1^* & 1 \\ iP_i^* & (i-1)P_{i-1}^* & (i-2)P_{i-2}^* & \cdots & 2P_2^* & P_1^* \end{vmatrix} \tag{4}$$

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