NOTE ON THE INFINITE DIVISIBILITY OF EXPONENTIAL MIXTURES

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1. Introduction. In one-counter waiting-time theory the Lindley case (cf. [2]) yields infinitely divisible stationary waiting-time distributions. In connection with this in the discussion to a paper by Kingman (cf. [4]) Runnenburg conjectured that the product of two independent exponentially distributed random variables is infinitely divisible. Goldie [1] proved that the product of two independent non-negative random variables is infinitely divisible if one of the two is exponentially distributed or, equivalently, that mixtures (with positive weights) of exponential random variables are infinitely divisible. In this note a slightly more general theorem is proved by a completely different method.

2. Definitions. We consider probability density functions (pdf's), which are mixtures of exponential pdf's, i.e. functions of the form

$$f(x) = \sum_{j=1}^{n} p_j \lambda_j e^{-\lambda_j x}, \tag{1}$$

where $p_j \neq 0$, $\sum_{j=1}^{n} p_j = 1$ and $\lambda_j > 0$. Without restriction we assume that the λ's are ordered in the following way

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n . \tag{2}$$

As $f(x)$ has to be non-negative it follows, by letting $x \to 0$ and $x \to \infty$ respectively, that $\sum_1^n p_j\lambda_j \geqq 0$ and $p_1 > 0$. This however is not sufficient for $f(x)$ to be positive as may be seen from the function $e^{-x} - 8e^{-2x} + 12e^{-3x}$, which is negative for $\log 2 < x < \log 6$. Simple sufficient conditions including negative p_j do not seem to be available.

The characteristic function (c.f.) corresponding to a pdf $f(x)$ will be denoted by $\phi(t)$. For definition and properties of infinitely divisible (inf div) c.f.'s we refer to Lukacs [3]. When convenient the pdf and the random variable corresponding to an inf div c.f. will also be called inf div.

3. A theorem.

Lemma 1. *A* c.f. $\phi(t)$ *is* inf div *if* $\log \phi(t)$ *can be expressed in the form*

$$\log \phi(t) = ita + \int_0^\infty [e^{itx} - 1 - itx/(1 + x^2)]g(x)\,dx, \tag{3}$$

where a is real, $g(x) \geqq 0$ and $[x^2/(1 + x^2)]g(x)$ integrable on $(0, \infty)$.

Proof. This is a special case of the Lévy-Khinchine representation (see [3]).

Lemma 2. *If* $f(x) = \lambda e^{-\lambda x}$ *then* $\phi(t) = \lambda/(\lambda - it)$ *and in the notation of Lemma* 1

$$a = \int_0^\infty (1 + x^2)^{-1} e^{-\lambda x}\,dx, \qquad g(x) = x^{-1}e^{-\lambda x}.$$

Proof. See [3].

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