TIMID PLAY IS OPTIMAL[1]

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You are in a gambling house Γ which is infinitely wealthy and offers all subfair bets, except

(a) it transacts business only in integer multiples of a dollar,

(b) it allows no credit, and

(c) money must be won or lost on each bet.

You start with a finite number of dollars, and keep betting until you go broke. How should you gamble so as to delay this fate as long as possible? Provided you have money, you should gamble next time so as to win or lose a dollar with probability $\frac{1}{2}$ each; that is, you should play timidly.

This theorem will be stated formally as Theorem 1, and proved below. A very similar result was obtained independently by Molenaar and van der Velde (1967). I first learned of the gambling house Γ while reading a draft of Leo Breiman's book for Addison-Wesley. Breiman showed that you do go broke.

Theorem 2 partially extends Theorem 1 to the continuous case. Theorem 3 is an analog of Theorem 1, with all bets uniformly subfair in a certain sense. Of course, Theorem 3 can be extended to the continuous case.

Let $X_1, X_2, \cdots$ be integer-valued random variables, $S_0 = 0$, $S_n = X_1 + \cdots + X_n$. Let j be a nonnegative integer. Say $j + S_n : n = 0, 1, \cdots$ is a *j-process* iff for all $n \geqq 0$: (i) $j + S_n \geqq 0$; (ii) $E(X_{n+1} \mid X_1, \cdots, X_n) \leqq 0$; (iii) $X_{n+1} \neq 0$ on $j + S_n \neq 0$. From (i) and (ii), $X_{n+1} = 0$ on $j + S_n = 0$. Informally, $j + S_n : n = 0, 1, \cdots$ is a possible process of fortunes if you gamble in Γ, starting with j (dollars). Let N_j be the least $n \geqq 0$ if any with $j + S_n = 0$; if none, $N_j = \infty$.

Theorem 1. *For nonnegative integer j and k, among all j-processes, $P(N_j > k)$ is maximized when: given $X_1, \cdots, X_n$, on $j + S_n > 0$, X_{n+1} is ± 1 with conditional probability $\frac{1}{2}$ each, for $0 \leqq n \leqq k - 1$.*

Corollary (Breiman). $P(N_j < \infty) = 1$.

The proof is easy, with the help of Lemmas 1, 2 and 3. A slightly more careful argument shows the maximum is strict. To state Lemma 1, let $u = \{u(n): n = 0, 1, \cdots\}$ be a sequence of real numbers. Define Tu, another sequence, as follows: $(Tu)(0) = 0$ and $(Tu)(n) = \frac{1}{2}u(n + 1) + \frac{1}{2}u(n - 1)$ for $n = 1, 2, \cdots$. Say u is *nice* iff $u(0) = 0$, $u(n)$ is nondecreasing with n, and $u(n + 1) - u(n)$ is nonincreasing with n.

Lemma 1. *If u is nice, so is Tu.*

Proof. Easy. ∎

To state Lemma 2, let $v(0) = 0$ and $v(n) = 1$ for $n \geqq 1$. Plainly, v is nice.

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