NOTES

HOW TO SURVIVE A FIXED NUMBER OF FAIR BETS[1]

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Suppose a gambler with initial capital b_0 wants to maximize his probability of still having a positive capital after n_0 successive independent bets, under two conditions: (a) the minimal stake is one dollar; (b) bets are fair and their probability of success is at most $\frac{1}{2}$.

A bet is determined by the stake c and the odds k: the gambler wins $kc - c$ with probability $1/k$ and loses c otherwise. If b_{m-1} denotes the gambler's capital after $m - 1$ bets, he must choose for the mth bet c_m $(1 \leqq c_m \leqq b_{m-1})$ and k_m $(k_m \geqq 2)$. For simplicity of presentation we make the inessential restriction that all b_m, c_m and k_m are integers. In a fair roulette (without zero) k can only be a divisor of 36. A bet $c = 1$, $k = 2$ is called conservative.

A situation is a pair (n, b) where b is the capital and n the number of bets to go. A strategy for (n_0, b_0) is a rule prescribing which bet should be made in the initial situation (n_0, b_0) and in each situation which may evolve from it. Under the stated conditions there exists for each (n_0, b_0) a (possibly non-unique) optimal strategy which leads to a (unique) maximal probability of survival (pos) denoted by $p(n_0, b_0)$. The independence of bets implies that for $n > 1$ and $b \geqq 1$

$$p(n, b) = \max_{c,k} \{(1/k)p(n - 1, b + kc - c) + (1 - 1/k)p(n - 1, b - c)\}.$$

Theorem 1. *The* pos $q(n, b)$ *for the conservative strategy* (i.e. $c = 1$ *and* $k = 2$ *in each situation*) *is for every* $n \geqq 1$ *a concave function of* b.

Proof. The theorem holds for $n = 1$ as $q(1, 0) = 0$, $q(1, 1) = \frac{1}{2}$ and $q(1, b) = 1$ for $b \geqq 2$. We proceed by induction. The definition of q implies that

$$q(n - 1, \beta) \geqq q(n, \beta) \tag{1}$$

and

$$q(n, b) = \tfrac{1}{2}q(n - 1, b + 1) + \tfrac{1}{2}q(n - 1, b - 1). \tag{2}$$

Substituting (1) with $\beta = b \pm 1$ into (2) we obtain $q(n, \lambda\beta_1 + (1 - \lambda)\beta_2) \geqq \lambda q(n, \beta_1) + (1 - \lambda)q(n, \beta_2)$, first for $\lambda = \frac{1}{2}$ and then by well known arguments for all $\lambda \in (0, 1)$ and all β_1, β_2 such that both sides of the inequality are defined.

Theorem 2. *The conservative strategy is optimal for all* n_0 *and* b_0.

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