THE CONDITIONAL LEVEL OF STUDENT'S t TEST[1]

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1. Introduction. Buehler and Fedderson (1963) considered the conditional significance level of Student's two-sided t-test and the coverage of the related confidence intervals. They conditioned on a subset of the form $|\bar{x}|/s \leqq c$ and found in one special case ($n = 2$, $\alpha = .5$) that for any values of μ_0 and σ^2 the conditional level of the t-test that the population mean is μ_0 is smaller than the unconditional level. In fact it is strictly smaller than a constant $a < \alpha = .5$. (For $c = \frac{2}{3}$ they were able to choose $a = .482$). Hence the conditional confidence coefficient of the confidence interval procedure is greater than $1 - a > .5$.

In this note we will show that similar results are valid for Student's two sided t-test at all levels and for all sample sizes, $n \geqq 2$. Also we show that the disparity between the conditional and unconditional levels is larger than was previously assumed. For example, in the case $n = 2$, $\alpha = .5$ we show that the conditional probability of acceptance given $|\bar{x}|/s \leqq \tan \pi/8 = 2^{\frac{1}{2}} + 1$ is bounded below by $\frac{2}{3}$.

In view of the well known optimum properties of the t-test it is not clear that the results of this note can possibly lead to any practically useful new procedures. (It is not even clear that any remotely reasonable test procedures exist for this problem which do not have conditional properties similar to those described here.)

We hope that these results about the t-test will help add to the general knowledge concerning its characteristics. In particular, let us point out that these results are somewhat related to the fact that the usual invariant estimator of σ is inadmissible (see Brown (to appear)). However it would appear that, if anything, these results concerning tests depend more strongly on normality than do the results for estimation.

2. Statement and proof of the main theorem. Let $X_1, X_2, \cdots, X_n$, $n \geqq 2$, be independent normal random variables with mean μ and variance σ^2. Let $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$ and $s^2 = n^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$. Let the (unconditional) rejection region for testing $\mu = \mu_0$ be of the form $K = \{\bar{x}, s: |\bar{x} - \mu_0|/s > k\}$. Then the level of significance, $\alpha = \Pr(K \mid \mu_0, \sigma)$, is independent of σ^2. Let the "conditioning" set be $C = \{\bar{x}, s: |\bar{x}|/s \leqq c\}$.

Theorem. *Suppose* $c > k/[(1 + k^2)^{\frac{1}{2}} - 1]$. *Then there is a constant* $a < \alpha$ *such that* $\Pr\{K \mid (\bar{x}, s) \in C, \mu_0, \sigma^2\} \leqq a < \alpha$ *for all* μ_0, σ^2.

Proof. Since K and C depend only on the ratios $\bar{x}/s$ and μ/s, and μ/σ, $\Pr\{K \mid (\bar{x}, s) \in C, \mu_0, \sigma\}$ is a function of the parameters only through the ratio

Received 1 December 1966.

[1] This research was supported in part by the Office of Naval Research Contract number Nonr 401(50) and in part by the National Science Foundation grant GP4867. Reproduction in whole or in part is permitted for any purpose of the United States Government.