EQUIVALENT GAUSSIAN MEASURES WITH A PARTICULARLY SIMPLE RADON-NIKODYM DERIVATIVE[1]

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1. Introduction. We consider two Gaussian probability measures P_ρ and P_r determined by covariance functions $\rho(s, t)$ and $r(s, t)$ respectively (the mean functions will be assumed to vanish). The well known Feldman-Hájek theorem asserts that P_ρ and P_r are either equivalent or perpendicular. If they are equivalent, the Radon-Nikodym derivative $(dP_\rho/dP_r)(x)$ exists and is the exponential of a quadratic form in x. This quadratic form may be diagonal, i.e., expressible as $\int_a^b f(t)x^2(t)\,dt$. If P_r is Wiener measure, L. A. Shepp [4], p. 352, has shown precisely when this happens. His method allows him to calculate $E\{\exp[-\frac{1}{2}\int_0^T f(t)x^2(t)\,dt]\}$ and this in turn permits him to prove an interesting zero-one law for the Wiener process. The purpose of this paper is to extend these results to an arbitrary Gaussian process.

We will use $r(s, t)$ consistently to denote a *continuous* covariance function defined on $[a, b] \times [a, b]$. For $f(t) \geqq 0$, we let $K(s, t) = [f(s)f(t)]^{\frac{1}{2}}r(s, t)$ which is then a positive (semi-definite) kernel and hence has nonnegative characteristic values [3], p. 237. Let λ_1 be the largest of these values. Finally, let $D(\lambda)$ and $K_\lambda(s, t)$ be the Fredholm determinant [3], p. 173, and resolvent kernel [3], pp. 151–158, corresponding to K.

Theorem 1. *Let $f(t)$ be nonnegative, bounded and measurable on $[a, b]$, and let $r(s, t)$, $D(\lambda)$ and λ_1 be as above. If $\lambda < 1/\lambda_1$, then*

$$E^r\{\exp[\tfrac{1}{2}\lambda \int_a^b f(t)x^2(t)\,dt]\} = [D(\lambda)]^{-\frac{1}{2}}.$$

Here $E^r\{\cdots\}$ denotes expectation on the Gaussian process with covariance function r.

Theorem 2. *Let $f(t)$ be positive and continuous on $[a, b]$ and let $r(s, t)$, $D(\lambda)$, $K_\lambda(s, t)$ and λ_1 be as above. If $\lambda < 1/\lambda_1$ and if we let $\rho_\lambda(s, t) = K_\lambda(s, t)/[f(s)f(t)]^{\frac{1}{2}}$, then $\rho_\lambda(s, t)$ is a covariance function, P_{ρ_λ} is equivalent to P_r and*

$$(1.1) \qquad (dP_{\rho_\lambda}/dP_r)(x) = [D(\lambda)]^{\frac{1}{2}} \exp[\tfrac{1}{2}\lambda \int_a^b f(t)x^2(t)\,dt].$$

Theorem 3 (zero-one law). *Let $f(t)$ be measurable on $[a, b]$ and let $r(s, t)$ be as above. The set of x's for which $f(t)x^2(t) \in L^1(a, b)$ is either of probability (P_r measure) one or zero, and these alternatives occur according as $f(t)r(t, t)$ is or is not in $L^1(a, b)$.*

If r is the covariance function of a stationary Gaussian process (i.e., $r(s, t) = p(|s - t|)$) so that $r(t, t)$ is a positive constant, we have the particularly simple

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